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An approach for consensual analysis on typical hesitant fuzzy sets via extended aggregations and fuzzy implications based on admissible orders

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RESUMO

MATZENAUER, Mônica Lorea. **Uma abordagem para análise consensual de conjuntos fuzzy hesitantes típicos via agregações estendidas e implicações fuzzy com base em ordens admissíveis**. Orientador: Renata Reiser. 2021. 107 f. Tese (Doutorado em Ciência da Computação) – CDTEC, Universidade Federal de Pelotas, Pelotas, 2021.

A Lógica Fuzzy Hesitante Típica (LFHT) está fundamentada na teoria dos Conjuntos Fuzzy Hesitantes Típicos (CFHT), os quais consideram como graus de pertinência os subconjuntos finitos e não vazios do intervalo unitário, chamados Elementos Fuzzy Hesitantes Típicos (EFHT). Nessa abordagem lógica, não apenas um número mas também subintervalos no intervalo unitário são também representações para EFHT, e podem ser aplicados no processo de tomada de decisão baseada em múltiplos critérios envolvendo muitos especialistas (TDMC-ME). Neste contexto, a LFHT provê a modelagem de situações onde ocorre não apenas incerteza de dados, mas também indecisão ou hesitação entre especialistas sobre os possíveis valores atribuídos às preferências referentes a coleções de objetos. Visando reduzir o colapso de informações para comparação e ranqueamento de alternativas nas relações de preferência, esta tese, primeiramente, desenvolve novas ideias sobre os conetivos lógicos da LFHT, as quais são investigadas no âmbito de três ordens admissíveis: (i) as ordens lexicográficas denominadas $\langle \mathbb{H}, \preceq_{Lex1} \rangle$ e $\langle \mathbb{H}, \preceq_{Lex2} \rangle$, relacionadas a ocorrência do menor/maior elemento em um CFHT ordenado de forma crescente e decrescente, respectivamente; (ii) a classe relevante das ordens $\langle \mathbb{H}, \preceq \rangle$, satisfazendo a propriedade de cardinalidade injectiva. Estudamos propriedades das negações e agregações, como as t-normas e operadores OWA são considerados, com especial interesse nas estruturas axiomáticas que definem as implicações e preservam suas propriedades algébricas e representabilidade. Estes estudos teóricos são aplicados a problemas TDMC-ME, para seleção de suporte a múltiplas alternativas de software. Como principal contribuição, introduzimos uma análise consenso sobre EFHT que formalmente contrói medidas de consenso por meio de funções de agregação estendidas, implicações e negações fuzzy. Usamos ordens admissíveis para comparação e, ainda, fornecendo uma análise de consistência sobre matrizes de preferência. A ação de automorfismos mostra-se oportuna para geração de novos operadores, preservando as principais propriedades consensuais que incluem unanimidade, consenso mínimo, dissensão máxima, simetria e invariância para replicação. O modelo $\mathcal{CC}_{\mathcal{AI}}$ aplica ordens admissíveis para promover o uso de medidas de consenso fuzzy, viabilizando comparações mesmo entre EFHT com cardinalidades diferentes. E ainda, o $\mathcal{CC}_{\mathcal{AI}}$-método é aplicado na análise consensual, via grupo de especialistas que consideram conjuntos fuzzy hesitantes típicos e fornecem classificações para múltiplos estilos de cervejas artesanais.

Palavras-chave: Conjuntos Fuzzy Hesitantes Típicos, Medidas de Consenso, Implicações Fuzzy, Agregações Estendidas, Ordens Admissíveis.

ABSTRACT

MATZENAUER, Mônica Lorea. **An approach for consensual analysis on Typical Hesitant Fuzzy Sets via extended aggregations and fuzzy implications based on admissible orders**. Advisor: Renata Reiser. 2021. 107 f. Thesis (Doctorate in Computer Science) – CDTEC, Federal University of Pelotas, Pelotas, 2021.

The Typical Hesitant Fuzzy Logic (THFL) is founded on the theory of the Typical Hesitant Fuzzy Sets (THFS), which are defined by considering as membership degrees the finite and non-empty subsets of the unit interval, which are called as Typical Hesitant Fuzzy Elements (THFE). In such logical approach, not only a number but also subintervals, in the unitary interval are also THFE-representations, which can be applied in the decision-making process based on multiple criteria involving many specialists (ME-MCDM). In this context, THFL provides the modelling for situations where there exists not only data uncertainty, but also indecision or hesitation among experts about the possible values for preferences regarding collections of objects. In order to reduce the information collapse for comparison and/or ranking of alternatives in the preference relationships, this thesis firstly develops new ideas about THFL's logical connectives, which are investigated within the scope of three admissible orders. In the set $\mathbb{H}$ of all hesitant fuzzy values, consider: (i) the lexicographic orders $\langle \mathbb{H}, \preceq_{Lex1} \rangle$ and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$, related to the occurrence of the smallest/largest element in an ascending/descending ordered THFS, respectively; (ii) the relevant class of order $\langle \mathbb{H}, \preceq_{A} \rangle$, satisfying the injective cardinality property. In particular, properties of negations and aggregations are studied, as t-norms and OWA operators, with special interest in the axiomatic structures defining the implications and preserving their algebraic properties and representability. Thus, these theoretical results are submitted to the ME-MCDM problem, in order to select the better support for multiple software alternatives. As a main contribution, in this thesis, we discuss consensus measures on THFE and present a model that formally builds consensus measures through extended aggregation functions and fuzzy negation, using admissible orders for comparison and further, differentiating an analysis of consistency over preference matrices. The action of automorphisms provides the generation of new conjugate operators, preserving the main consensual properties as proposed in the Beliakov's research, including unanimity, minimum consensus, maximum dissension, symmetry and invariance for replication. The new $\mathcal{CC}_{\mathcal{AI}}$-method is presented, by applying admissible orders and promoting the use of fuzzy consensus measures based on multi-valued fuzzy logics, and, then, this work enables comparisons even between THFE with different cardinalities. These new theoretical results are then applied to another ME-MCDM problem, obtaining $\mathcal{CC}_{\mathcal{AI}}$-consensus in a group of experts which consider typical hesitant fuzzy sets to provide classifications for multiple styles of craft beers.

Keywords: Typical Hesitant Fuzzy Sets, Consensus Measures, Fuzzy Implications, Extended Aggregations, Admissible Orders.

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LIST OF ABBREVIATIONS AND ACRONYMS

AF	*Aggregation Function*
AHFA	*Adjusted HFA*
AHFWA	*Adjusted HFWA*
CRP	*Consensus Reaching Process*
DM	*Decision Making*
EAF	*Extended Aggregation Function*
FPR	*Fuzzy Preference Relations*
FPR-SC	*Fuzzy Preference Relations with Self-Confidence*
FS	*Fuzzy Sets*
GDM	*Group Decision Making*
HFA	*Hesitant Fuzzy Averaging*
HFE	*Hesitant Fuzzy Elements*
HFLPR	*Hesitant Fuzzy Linguistic Preference Relations*
HFLTS	*Hesitant Fuzzy Linguistic Term Sets*
HFPR	*Hesitant Fuzzy Preference Relations*
HFS	*Hesitant Fuzzy Sets*
HPFPR	*Hesitant Probabilistic Fuzzy Preference Relations*
HFPWA	*Hesitant Fuzzy Prioritized Weighted Average*
HFWA	*Hesitant Fuzzy Weighted Average*
IFS	*Intuitionistic Fuzzy Sets*
IHFOW	*Induced Hesitant Fuzzy OWA*
IOWA	*Induced Ordered Weighted Average*
IVFS	*Interval-Valued Fuzzy Sets*
IVHF	*Interval-Valued Hesitant Fuzzy*
LSGDM	*Large-Scale Group Decision Making*
MAGDM	*Multi-Attribute Group Decision Making*
MCDM	*Multi-Criteria and Decision Making*
ME-MCDM	*Multi Expert-Multi Criteria Decision Making*

MG-DTRS	*Multi-Granulation Decision Theoretic Rough Sets*
OWA	*Ordered Weighted Average*
PLOWA	*Probabilistic linguistic OWA*
PLPR	*Probabilistic Linguistic Preference Relation*
PLTS	*Probabilistic Linguistic Term Sets*
PLWA	*Probabilistic Linguistic Weighted Average*
SVFS	*Set-Valued Fuzzy Sets*
T2FS	*Type-2 Fuzzy Sets*
THAF	*Typical Hesitant Aggregation Functions*
THEA	*Typical Hesitant Extended Aggregation*
THFC	*Typical Hesitant Fuzzy Co-Implications*
THFE	*Typical Hesitant Fuzzy Elements*
THFI	*Typical Hesitant Fuzzy Implications*
THFL	*Typical Hesitant Fuzzy Logic*
THFN	*Typical Hesitant Fuzzy Negations*
THFPR	*Typical Hesitant Fuzzy Preference Relation*
THFS	*Typical Hesitant Fuzzy Sets*
WA	*Weighted Average*

CONTENTS

1 INTRODUCTION

Fuzzy Set Theory (FS), presented by (ZADEH, 1965), has yielded several extensions over the years. Among the most relevant contributions in this research area, we highlight the following works:

- the extension known as Type-2 Fuzzy Sets (T2FS), introduced in (ZADEH, 1971, 1975), which considers the membership functions as FS on the unit interval $[0, 1]$;
- the Set-Valued Fuzzy Sets (SVFS), introduced by (GRATTAN-GUINNESS, 1976) expressing the membership degrees as subsets of the unit interval $[0, 1]$;
- Atanassov's Intuitionistic Fuzzy Sets (IFS) in (ATANASSOV, 1986), where the definition of a fuzzy set considers not only the membership function, but also its dual construction providing the non-membership degree; and
- Hesitant Fuzzy Sets (HFS), proposed by (TORRA; NARUKAWA, 2009) as another extension for fuzzy sets, in which the membership degree of an hesitant fuzzy set is also given as a subset of $[0, 1]$.

Note that despite the existence of different extensions in order to handle imprecise information, there are relationships among them as showed in (BUSTINCE et al., 2016). This work explored the inclusion relationships between some types of fuzzy sets and it also concluded that, in fact, the concepts of HFS and SVFS are equivalent. However, the results achieved in (TORRA, 2010) presented an explicit definition for the union and intersection operations on HFS, which was not the research focused in (GRATTAN-GUINNESS, 1976).

In sequence, (BEDREGAL et al., 2014a) noticed that in most applications of HFS, Typical Hesitant Fuzzy Elements (THFE) are used, i.e. finite and non-empty hesitant fuzzy degrees. Then, the notion of Typical Hesitant Fuzzy Logic (THFL) appears, which is based on Typical Hesitant Fuzzy Sets (THFS) conceived taking THFE as membership degrees.

Relevant research in decision making has been supported by the HFS theory, since it was introduced in 2010. See, for instance, the studies in (ZENG et al., 2020;

FARHADINIA; AICKELIN; KHORSHIDI, 2020; REZAEI; REZAEI, 2020; FARHADINIA; XU, 2020; WANG et al., 2021) considering the logical study of HFS. In particular, several weighted average (WA) and ordered weighted average (OWA)-like operators have been proposed to be used in multi-criteria and decision making (MCDM) problems dealing with multiple attributes and multiple specialists (BEDREGAL et al., 2014a; XIA; XU, 2011a; ZHU; XU; XIA, 2012; MATZENAUER et al., 2021).

A frequent issue in the context of MCDM problems is that it is not always possible to find a consensus among a group of experts. So, it seems more appropriate to consider a set of possible values taking into account everyone's opinion. For instance, in order to provide a membership degree for an element of the universe, HFS can be useful to express this membership degree through a set of THFE, which will consider all the opinions given by the group of experts.

However, some research questions arise from this setting:

(i) How much do these elements agree with each other?

(ii) Is it possible to combine these elements into a single output?

(iii) Is the result reliable and does it reflect the opinions provided by the group?

The consolidated research on consensus measures provide relevant results contributing to answer all these questions, which have been applied in different contexts. In the current literature, we can find works on fields like consensual processes (UNZU; VORSATZ, 2011), consensual measures and aggregations (BELIAKOV; CALVO; JAMES, 2014), majority decisions (LAPRESTA; LLAMAZARES, 2010) and preference intensities group decision and negotiation (GARCÍA-LAPRESTA; LLAMAZARES, 2001; LLAMAZARES; PÉREZ-ASURMENDI; GARCÍA-LAPRESTA, 2013).

Apart from other results, we provide a general idea of up to what extent the expert inputs agree with one another based on our approach using the theory of THFS. In our proposal, we present a model that formally constructs consensus measures by means of aggregations functions, fuzzy implication-like functions and fuzzy negations, using admissible orders to compare the THFE, and also providing an analysis of consistency on them. Our theoretical results are applied into a problem of decision making with multi-criteria illustrating our methodology to achieve consensus in a group of experts working with typical hesitant fuzzy sets.

1.1 Main contributions

As the main contribution, this thesis introduces a model to provide semantic interpretation for consensus setting on Typical Hesitant Fuzzy Sets, namely the

$CC_{\mathbb{H}}$-Model: Consensus Measures on Typical Hesitant Fuzzy Sets (THFS) based on Extended Aggregation and Implication Operators.

The main properties proposed in the literature of consensus measures are studied here from the setting of inputs on the class of THFS defined over $\mathbb{H}$, which is the set of all finite and non-empty subsets of the unit interval $[0, 1]$, mainly according to the approach in (BELIAKOV; CALVO; JAMES, 2014). However, the studies are restricted to the fact that the agreement among evaluations has the same relevance.

Going beyond, among several partial orders defined over $\mathbb{H}$, the present proposal represents an extension that considers an admissible partial order $\leq_A$ on $\mathbb{H}$; based on an extended aggregation operator A, as a binary relation promoting comparison even between THFE with different cardinalities.

The main contributions achieved with the development of this work are listed below:

(i) Starting with fuzzy extensions of consensus measures from U to $\mathbb{H}$ and taking into account so many distinct partial orders for THFS, this work introduces a new admissible order based on a hesitant aggregation function $\mathcal{A}$, refining not only the restricted consensual order $<_{RH}$ but many other ones, providing a comparison between HFS which do not have the same cardinalities;

(ii) This research considers the concepts of admissible orders obtained from hesitant aggregation functions and fuzzy negations, providing methods to generate comparisons (ordering) of typical hesitant fuzzy elements (BUSTINCE et al., 2013; MIGUEL et al., 2016; LIMA, 2019);

(iii) Extension of the main concepts of fuzzy connectives (fuzzy negations, aggregation functions, t-norms and t-conorms, fuzzy implications) are considered, discussing their properties regarding admissible orders (BUSTINCE et al., 2020; MATZENAUER et al., 2021);

(iv) This work also considers a formal definition of consensus measures extending this concept to the typical hesitant fuzzy sets. Actually, various applications can be found in the literature concerning consensus measures (LI et al., 2018; RODRÍGUEZ et al., 2018), and also an attempt of a formal mathematical definition was given in (BELIAKOV; CALVO; JAMES, 2014).

(v) This study explores methodologies based on the $CC_{\mathbb{H}}$-Model, a construction of consensus measures through different implications, exploring main properties of fuzzy implications which are required, regarding admissible/total orders on $\mathbb{H}$.

1.2 Objectives

The main objective of this thesis is to extend Beliakov's work by applying consensus measures to typical hesitant fuzzy sets based on aggregation functions and admissible orders.

The specific objectives are described as follows:

- Contribute for the study of hesitant fuzzy sets and admissible linear orders, considering their relevance in multi-valued fuzzy sets by allowing comparison and ordering relations;
- Collaborate to the study of hesitant fuzzy aggregators, exploring the main properties, analysing their relevance to consensus measurement methodologies and practical applications;
- Study fuzzy consensus measures and provide the application of related methodology in the decision making based on multi-criteria and -attribute from many specialists;
- Introduce the study of main classes of hesitant fuzzy implications, exploring the main properties and constructors of duality and conjugation;
- Propose the $CC_{\mathbb{H}}$-Model, based on an axiomatic definition of consensus measures consistent with studies of admissible linear orders in typical hesitant fuzzy sets.

1.3 Outline of the thesis

This work is organized as follows. After this introductory chapter, the related works are presented in Chapter 2.

Chapter 3 presents the definition of aggregation function, including t-norms, t-conorms, and also other fuzzy connectives. Besides, partial orders are approached on hesitant fuzzy sets and the notion of consensus measures is also reported (BELIAKOV; CALVO; JAMES, 2014).

Next, in Chapter 4, it is presented a new Admissible $\mathbb{H}$ Order for the HFS elements. And this allows us to introduce some operators, such as the typical hesitant fuzzy negations and typical hesitant t-norms.

In Chapter 5, the notion of some typical hesitant fuzzy connectives is presented, based on an admissible order $\preceq$ related to the poset $(\mathbb{H}, \leq)$. Main properties of $\langle\mathbb{H}, \preceq\rangle$-negations and $\langle\mathbb{H}, \preceq\rangle$-implication functions are analysed.

Chapter 6 presents the notion of $\langle\mathbb{H}, \preceq\rangle$-implications, as typical hesitant fuzzy implications considering admissible $\langle\mathbb{H}, \preceq\rangle$-orders, discussing their main properties.

Chapter 7 describes the proposed strategy to solve an ME-MCDM (Multi Expert-Multi Criteria Decision Making) problem taking into account the admissible $\langle \mathbb{H}, \preceq \rangle$-orders introduced in previous chapters.

In Chapter 8 we extend the notion of consensus measures on Typical Hesitant Fuzzy Sets. Based on the formal definition of a consensus measure on the $\mathbb{H}$, we formalise $\mathcal{CC}_{\mathcal{A},\mathcal{I}}$-Models to obtain new methodologies of consensus preserving main properties in the context of Typical Hesitant Fuzzy Sets. This study also considers the corresponding extensions of aggregations, implications and fuzzy negations.

Finally, the Final Considerations presents the results corresponding the publications achieved so far.

2 RELATED WORKS

This chapter presents a study of related works on several tutorials and reviews that emphasize the relevance of the investigation on consensus measures, as presented in (BELIAKOV; CALVO; JAMES, 2014) and (PRADERA et al., 2016).

Here, we highlight works that, besides being the state of the art, present relevant results and analyze them with emphasis on the following applied concepts: hesitant fuzzy sets, aggregation operators and consensus measures. This selection also considers the applied research field.

Thus, the following list highlights the author and publication year, summarizing the main characteristics of each work. Table 2 gathers these relevant articles in the area, showing the paper title according to the list, the aggregation operators (since these operators are used to compare the results) and/or strategy used, and the applied field.

The selected articles are briefly described as follows.

1. In paper (XU; CABRERIZO; HERRERA-VIEDMA, 2017) an additive consistency based estimation measure to normalize the HFPR is proposed, based on which, a consensus model is developed. Here, two feedback mechanisms are proposed, namely, interactive mechanism and automatic mechanism, to obtain a solution with desired consistency and consensus levels, using induced ordered weighted averaging (IOWA) operator to aggregate the individual HFPR into a collective one.

2. The results in (WU; XU, 2018) introduce LSGDM consensus model in which the clusters are allowed to change, and the decision makers provide preferences using fuzzy preference relations. A novel distance measure over the possibility distribution based on the hesitant fuzzy element is given to compute the various consensus measures, after which a local feedback strategy, with four identification rules and two direction rules, is designed to guide the consensus reaching process.

3. In the study of (ZHANG; WANG; TIAN, 2015), it is developed a decision support model that simultaneously addresses the consistency and consensus for group decision making based on hesitant fuzzy preference relations. Two

convergent algorithms were proposed, using hesitant fuzzy averaging operator in the developed support model.

4. In (RODRÍGUEZ et al., 2018), the paper presents a new adaptive consensus reaching processes (CRP) model to deal with large-scale group decision making (LSGDM) which includes: clustering process to weight expert's sub-groups taking into account their size and cohesion, the use of hesitant fuzzy sets to fuse expert's sub-group preferences to keep as much information as possible, and the definition of an adaptive feedback process that generates advice depending on the consensus level achieved to reduce the time and supervision costs of the CRP.

5. In the paper (GARCÍA-LAPRESTA; PÉREZ-ROMÁN, 2016), an agglomerative hierarchical clustering process is proposed, where the clusters of agents are generated by using a distance-based consensus measure; considering that agents judge the feasible alternatives through linguistic terms – when they are confident in their opinions – or linguistic expressions formed by several linguistic terms when they hesitate.

6. The consistency of aggregations defined as HPFPR in (BASHIR; RASHID; IQBAL, 2018) provides novel algorithms, achieving reasonable consensus between decision makers. The final algorithm proposed comprehends other algorithms shown, presenting a complete decision support model for group decision making.

7. The study in (ZHANG; LI; LIANG, 2020) presents, after reviewing the relevant literature, four kinds of interval-valued hesitant fuzzy (IVHF) multi-granulation decision-theoretic rough sets (MG-DTRS) over two universes proposed according to different risk appetites of experts. Then, they explore some fundamental propositions of newly proposed models.

8. In the paper (WU; XU, 2016), the authors develop separate consistency and consensus processes to deal with HFLPR individual rationality and group rationality. First, a possibility distribution approach and a 2-tuple linguistic model are introduced as support tools. Then, a new consistency measure is defined and a convergent algorithm described to aid the consistency improvement process in a given HFLPR.

9. In (LAN et al., 2018), it is proposed a new order relation extraction method based on a new additive consistency fuzzy preference relation for hesitant fuzzy elements (HFE). Then, the proposed additive consistency fuzzy preference relation is applied to integrate group decision information.

10. The study of (LIAO; XU; XIA, 2014) introduces the concepts of multiplicative consistency, perfect multiplicative consistency and acceptable multiplicative consistency for a hesitant fuzzy preference relation, based on which, two algorithms are given to improve the inconsistency level of a hesitant fuzzy preference relation. Furthermore, the consensus of group decision making is studied based on the hesitant fuzzy preference relations.

11. The paper (LIU; JIANG, 2019) presents some novel consistency and consensus improvement methods by using an optimization technique proposed for three actual cases: (1) the decision makers refuse to modify their opinions and the problem is without time pressure; (2) the decision makers are willing to modify their opinions and the problem is without time pressure; (3) the decision-making problem is under time pressure.

12. In (ZHANG; LIANG; ZHANG, 2018), it is defined a new distance measure for two HFLTS and it is proposed a distance-based consensus measure for the MAGDM with HFLTS. Then, based on this consensus measure, they develop a minimum adjustment distance consensus rule for the MAGDM with HFLTS, which can minimize the adjustment distance between the original and adjusted opinions in the process of reaching consensus. Moreover, to obtain the collective opinion with maximum consensus, it is developed a minimum distance aggregation model, which minimizes the maximum of the distance between each decision maker's individual opinion and the collective opinion. Furthermore, based on the proposed consensus rule and aggregation model, it is presented a consensus reaching process for MAGDM with HFLTS.

Paper Title	Operator / Strategy	Applied Field
1. A consensus model for hesitant fuzzy preference relations and its application in water allocation management	IHFOWA	GDM
2. A consensus model for large-scale group decision making with hesitant fuzzy information and changeable cluster	WA	LSGDM
3. A decision support model for group decision making with hesitant fuzzy preference relations	HFA	GDM with HFPR
4. A large scale consensus reaching process managing group hesitation	HFWA	LSGDM

5. Consensus-based clustering under hesitant qualitative assessments	OWA, arithmetic mean and median	clustering process
6. Hesitant probabilistic fuzzy preference relations in decision making	HPFA	GDM
7. Interval-valued hesitant fuzzy multi-granularity three-way decisions in consensus processes with applications to multi-attribute group decision making	optimistic and pessimistic tactics and IVHF hybrid averaging	MAGDM
8. Managing consistency and consensus in group decision making with hesitant fuzzy linguistic preference relations	WA and arithmetic average	HFLPR
9. Multi-attribute group decision making based on hesitant fuzzy sets, TOPSIS method and fuzzy preference relations	TOPSIS method	MAGDM
10. Multiplicative consistency of hesitant fuzzy preference relation and its application in group decision making	AHFWA and AHFA	GDM with HFP information
11. Optimizing consistency and consensus improvement process for hesitant fuzzy linguistic preference relations and the application in group decision making	WA	HFLPR in GDM
12. Reaching a consensus with minimum adjustment in MAGDM with hesitant fuzzy linguistic term sets	minimum distance aggregation model	MAGDM with HFLTS

Table 1 – The use of aggregators among the main papers analyzed about consensus measure in hesitant fuzzy sets.

So, through this study of the related works on the hesitant fuzzy sets and consensus measures research areas, it is verified that none of these articles had made use of implication functions for the construction of the methods defining consensus measures.

Moreover, despite considering a restrict universe under which the applications are developed, the selected research papers show that the formal definition of consensus measures based on THFS considering the ordered structure provided by admissible linear orders is a prospective research field.

Thus, in this thesis we discuss consensus measures for THFE, which are the finite and non-empty fuzzy membership degrees under the scope of THFS. Also, we present a model that formally constructs consensus measures by means of aggregations functions, fuzzy implication-like functions and fuzzy negations, using admissible orders to compare the THFE. Then, theoretical results are applied into a problem of decision making with multi-criteria, illustrating a methodology to achieve consensus in a group of experts working with typical hesitant fuzzy sets.

3 PRELIMINARIES

In this chapter, basic concepts of aggregation functions on the unit interval $[0,1]$ are revisited, and also some properties are recalled and exemplifications are pointed out (BELIAKOV; BUSTINCE; CALVO, 2016; GRABISCH et al., 2009). We also include the important class of triangular norms (GRABISCH et al., 2009; KLEMENT; MESIAR; PAP, 2000).

In addition, fuzzy connectives, as negations and implications are also considered and their main properties are reported (ALSINA; MAURICE; SCHWEIZER, 2006; BELIAKOV; PRADERA; CALVO, 2007).

3.1 Concepts on fuzzy set theory

Extended fuzzy aggregation operators, partial orders on fuzzy set theory and fuzzy connectives are considered in this section.

3.1.1 Partial orders on fuzzy sets

Let us recall the notions of partial ordering. Let P be a non-empty set, a partial order $\leq$ on the set P is a binary relation on P which satisfies:

P1: $p \leq p$, for each $p \in P$ (reflexivity),

P2: If $p \leq q$ and $q \leq p$, then $p = q$ for all $p, q \in P$ (antisymmetry),

P3: If $p \leq q$ and $q \leq r$, then $p \leq r$ for all $p, q, r \in P$ (transitivity).

Observe that we will say $a < b$ when (a, b) is in a relation $\leq$ but $a \neq b$. A set P with a partial order $\leq$ is referred to as a partially ordered set (poset) and denoted by $(P, \leq)$. If any two elements a, b are comparable in a poset $(P, \leq)$, i.e. either $a \leq b$ or $b \leq a$, then the partial order $\leq$ is said to be a linear order (and then P is a chain).

In this work, we consider the poset $([0,1], \leq)$, where $\leq$ is the partial order of real numbers restricted to unit interval $[0,1]$.

3.1.2 Fuzzy aggregation operators

Based on (BELIAKOV; PRADERA; CALVO, 2007), main properties of aggregation functions are reported in the following.

Definition 3.1.1 *A function* $A : [0,1]^n \to [0,1]$ *is an* n*-ary aggregation function (AF) if it verifies the following conditions*

A1: If $x_i \leq y_i$*, for each* $i = 1, \ldots, n$*, then* $A(x_1, \ldots, x_n) \leq A(y_1, \ldots, y_n)$ *(isotonicity);*

A2: $A(0, \ldots, 0) = 0$ *and* $A(1, \ldots, 1) = 1$ *(boundary conditions).*

Consider $\mathbb{N}_n = \{1, 2, \ldots, n\}$, for a natural number n such that $n \geq 1$. Other properties which can be required for an aggregation function A:

A3: $A(x_1, \ldots, x_n) = A(x_{(1)}, \ldots, x_{(n)})$ for each permutation $(\,) : \mathbb{N}_n \to \mathbb{N}_n$ (symmetry).

An aggregation function A is a disjunctive (conjunctive) function if, respectively, we have that:

A4: $A(1, x) = A(x, 1) = 1$, for all $x \in [0,1]$;

A5: $A(0, x) = A(x, 0) = 0$, for all $x \in [0,1]$.

Definition 3.1.2 *A function* $A\colon \bigcup_{n=1}^{\infty} [0,1]^n \to [0,1]$ *is an extended aggregation (EAF) if the following condition holds:*

$$\forall n \in \mathbb{N}_2,\ A \restriction [0,1]^n : [0,1]^n \to [0,1] \text{ is an AF and } A(x) = x,\ \forall x \in [0,1].$$

Thus, based on Definition 3.1.2, we can identify any EAF A with a family of functions $(A_n)_{n \in \mathbb{N}_2}$ such that each $A_n : [0,1]^n \to [0,1]$ is an AF.

For EAF, the following two properties can also be considered:

A6: $A(x_1, \ldots, x_n) = A(x_1, \ldots, x_n, \ldots, x_1, \ldots, x_n)$ (invariance for replications);

A7: $A(x_1, \ldots, x_n) = A(x_1, \ldots, x_i, 1, x_{i+1}, \ldots, x_n)$ (invariance for 1).

According with (BELIAKOV; PRADERA; CALVO, 2007), generally, a given property holds for an EAF A if, and only if, it holds for each member of the family A_n.

Remark 3.1.1 *Given a bivariate aggregation function* A*, we can define an EAF* $A'\colon \bigcup_{n=1}^{\infty} [0,1]^n \to [0,1]$*, as follows:*

$$A'(x) = x \quad \textit{and} \quad A'(x_1, \ldots, x_n) = A(x_1, A'(x_2, \ldots, x_n)), \forall n \in \mathbb{N}_2. \tag{1}$$

Proposition 3.1.1 *(YAGER, 1988) Let* $\omega = (w_1, w_2, \dots w_n) \in [0,1]^n$ *be a positive weighting vector, meaning that its components are all non-negative (*$w_i \in [0,1]$*) and their sum equals one, i.e.* $\sum_{i=1}^n w_i = 1$*. Let* $(x_1, \dots, x_n) \in [0,1]^n$ *and* $\sigma \colon \mathbb{N}_n \to \mathbb{N}_n$*, with* $\mathbb{N}_n = \{1, 2, \dots, n\}$ *be a permutation such that* $x_{\sigma(1)} \geq \dots \geq x_{\sigma(n)}$*, then an* n*-dimensional ordered weighted averaging (*OWA_ω*) operator w.r.t. a weighting vector* ω *is a function* $OWA_\omega \colon [0,1]^n \to [0,1]$*, defined by:*

$$OWA_\omega(x_1, \dots, x_n) = \sum_{i=1}^{n} w_i x_{\sigma(i)}, \tag{2}$$

verifying A1, A2 and A3 properties and also the idempotency property:

(A8) $A(x, \dots, x) = x$*, for all* $x \in [0,1]$*.*

The OWA operators provide a parameterized family of aggregation operators, which include many of the well-known operators such as the maximum, the minimum, the median and the arithmetic mean. In order to obtain these particular operators, we should simply choose particular weights.

According to (BELIAKOV; PRADERA; CALVO, 2007, Section 2.5.2), an OWA is a non-decreasing (strictly increasing if all weights are positive) and idempotent; OWA functions are continuous, symmetric, homogeneous and shift-invariant; OWA functions do not have neutral or absorbing elements, except for the minimum and maximum.

Proposition 3.1.2 *Let* $OWA_\omega \colon [0,1]^n \to [0,1]$ *be the ordered weighted averaging operator, as given in Eq.(2). The extended ordered weighted averaging operator* $OWA'_\omega \colon \bigcup_{n=1}^{\infty} [0,1]^n \to [0,1]$*, obtained by Eq.(1) and expressed as follows*

$$OWA'_\omega(x_1, \dots, x_n) = OWA_\omega(x_1, OWA'_\omega(x_2, \dots, x_n)), \forall n \geq 2, \tag{3}$$

verifies A1, A2, A3 and A8 properties.

Proof: Straightforward from Proposition 3.1.1. □

The definition of triangular norms is presented next. Since each t-norm is an aggregation function with four specific properties, it enables a formal interpretation of intersections in the theory of fuzzy sets.

Definition 3.1.3 *A function* $T \colon [0,1]^2 \to [0,1]$ *is a t-norm if, for each* $x, y, z \in [0,1]$*, it satisfies:*

T1: $T(x,y) = T(y,x)$ *(commutativity);*

T2: $T(x, T(y,z)) = T(T(x,y), z)$ *(associativity);*

T3: If $x \leq y$*, then* $T(x,z) \leq T(y,z)$ *(isotonicity);*

T4: $T(x,1) = x$ *(neutrality of* 1*-element).*

3.1.3 Other fuzzy connectives

Definition 3.1.4 *A function $N : [0,1] \to [0,1]$ is a fuzzy negation if*

N1: $N(0) = 1$ *and* $N(1) = 0$*;*

N2: If $x \leq y$ *then* $N(y) \leq N(x)$*, for all* $x, y \in [0,1]$*.*

A fuzzy negation N is strict if it is continuous additionally, it is strong if it is involutive, i.e.

N3: $N(N(x)) = x, \forall x \in [0,1]$*;*

N4: If $x < y$*, then* $N(y) < N(x)$*.*

The most common strong fuzzy negation is $N_S(x) = 1 - x$, also known as the standard or Zadeh negation. Each strong fuzzy negation is strict, but the converse does not hold. For example, the negation $N(x) = 1 - \sqrt{x}$ is strict, but it is not strong.

Moreover, the action of a strong fuzzy negation N on a function $f\colon [0,1]^n \to [0,1]$, denoted by f_N and named the N-dual function of f, is defined as:

$$f_N(\vec{x}) = (f(N(\vec{x}), N(\vec{y}))), \vec{x}, \vec{y} \in [0,1]^n. \quad (4)$$

We now recall the notion of (fuzzy) implication function. An implication function, in the sense of Fodor and Roubens, see (BACZYŃSKI et al., 2013; BACZYŃSKI; JAYARAM, 2008; BUSTINCE; BURILLO; SORIA, 2003; PRADERA et al., 2016), is a mapping $I\colon [0,1]^2 \to [0,1]$, such that, for every $x, y, z \in [0,1]$:

I1: If $x \leq y$, then $I(y,z) \leq I(x,z)$ (first place antitonicity);

I2: If $y \leq z$, then $I(x,y) \leq I(x,z)$ (second place isotonicity);

I3: $I(0,0) = 1$ (left boundary);

I4: $I(1,1) = 1$ (right boundary);

I5: $I(1,0) = 0$ (corner condition).

Other properties may be demanded from these implication functions, depending most of the times on the application, see (BACZYŃSKI; JAYARAM, 2008), for instance. A brief list is given as follows:

I6: $I(0,y) = 1$ (left boundary);

I7: $I(x,1) = 1$ (right boundary);

I8: $I(x,x) = 1$ (identity principle);

I9: $I(x, I(y, z)) = I(y, I(x, z))$ (exchange principle);

I10: $I(x, y) = I(N(y), N(x))$, for a strong negation N (contrapositive);

I11: If $x \leq y$, then $I(x, y) = 1$ (left-ordering property);

I12: If $I(x, y) = 1$, then $x \leq y$ (right-ordering property).

Table 2 reports the definition of fuzzy implications and corresponding N_S-dual coimplications considered in this work. In some works, such as (BUSTINCE; BURILLO; SORIA, 2003; SHI et al., 2010), the properties and the relations of such fuzzy connectives have been investigated.

Table 2 – Fuzzy (Co)Implications.

Fuzzy Implications	Fuzzy Coimplications
$I_{FD}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ \max(1-x, y), & \text{otherwise.} \end{cases}$	$J_{FD}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ \min(1-x, y), & \text{otherwise.} \end{cases}$
$I_{LK}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 1-x+y, & \text{otherwise.} \end{cases}$	$J_{LK}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ y-x, & \text{otherwise.} \end{cases}$
$I_{RB}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 1-x+xy, & \text{otherwise.} \end{cases}$	$J_{RB}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ y-xy, & \text{otherwise.} \end{cases}$
$I_{30}(x,y)=\begin{cases} \min(1-x, y, 0.5), & \text{if } 0<y<x<1, \\ \min(1-x, y), & \text{otherwise.} \end{cases}$	$J_{30}(x,y)=\begin{cases} \max(1-x, y, 0.5), & \text{if } 0<x<y<1, \\ \max(1-x, y), & \text{otherwise.} \end{cases}$
$I_{RS}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise.} \end{cases}$	$J_{RS}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ 1, & \text{otherwise.} \end{cases}$
$I_{GD}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ y, & \text{otherwise.} \end{cases}$	$J_{GD}(x,y)=\begin{cases} y, & \text{if } x \geq y, \\ 1, & \text{otherwise.} \end{cases}$
$I_{WB}(x,y)=\begin{cases} 1, & \text{if } x \neq y, \\ y, & \text{otherwise.} \end{cases}$	$J_{WB}(x,y)=\begin{cases} y, & \text{if } x \neq y, \\ 1, & \text{otherwise.} \end{cases}$
$I_{GG}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{otherwise.} \end{cases}$	$J_{GG}(x,y)=\begin{cases} 0, & \text{if } x \leq y, \\ \frac{y-x}{1-x}, & \text{otherwise.} \end{cases}$
$I_{YG}(x,y)=\begin{cases} 1, & \text{if } x = 0 \text{ and } y = 0, \\ y^x, & \text{otherwise.} \end{cases}$	$J_{YG}(x,y)=\begin{cases} 0, & \text{if } x = 1 \text{ and } y = 1, \\ 1-(1-x)^{(1-y)}, & \text{otherwise.} \end{cases}$

3.2 Concepts on fuzzy consensus measures

First, the notion of fuzzy preference relation is considered and thus, the axiomatic definition of fuzzy consensus measures is reported.

Multi-criteria decision making (MCDM) refers to evaluating, prioritizing or selecting over some available alternatives $\{A_1, A_2, ..., A_p\}$ with respect to a set of criteria $\{c_1, c_2, ..., c_q\}$, which are usually conflicted with each other. In order to do that, it is necessary to assign a value to each alternative, with respect to each criterion.

For instance, suppose some friends provide ratings for three styles of craft beers (shown in Table 3). As well as comparing the average, we can observe that while everyone partially agrees that the Sour style craft beer is not very good, and the Pale Ale style is not too bad, there is a lack of consensus regarding the Weiss style.

Table 3 – Individual ratings for craft beer styles.

Craft Beer Style	F1	F2	F3	F4	Average
Sour (S)	0.3	0.2	0.2	0.4	0.275
Weiss (W)	0.2	0.3	0.9	0.8	0.55
Pale Ale (P)	0.65	0.7	0.65	0.6	0.65

Consensus measures, that is, functions which give an entirely idea of how much the inputs agree with one another, have been increasingly employed in decision making contexts. Such measures have been used in voting and preference aggregation, for example to describe a set of voters and group them according to the similarity in their preferences (BELIAKOV; CALVO; JAMES, 2014).

Similar to the standard divergence of the mean in statistical summaries, consensus measures can provide an indication of reliability or the degree to which an entire evaluation reflects the opinions of all individuals in a group. As such, they have also been used to inform consensus reaching processes, where a minimum level of consensus can be set and a final decision may not be accepted if the consensus measure output is below a threshold. The consensus level between the pairwise preferences of an individual and the group can also be used to make recommendations increasing the overall agreement between experts (BELIAKOV; CALVO; JAMES, 2014).

For a finite set of objectives, $\chi = \{x_1, x_2 ..., x_n\}$, a fuzzy preference relation (FPR) is defined by a fuzzy set on the product set $\chi \times \chi$, as follows.

Definition 3.2.1 *(B. ZHU, 2014, Definition 1) A FPR* $\mathrm{R} \subseteq \chi \times \chi$ *is characterized by the membership function* $\mu_{\mathrm{R}} : \chi \times \chi \rightarrow [0, 1]$.

Thus, the FPR R can be represented by an $n \times n$ matrix $R = (r_{ij})_{n \times n}$, where $r_{ij} = \mu_{\mathrm{R}}(x_i, x_j)$, for all $i, j \in \{1, 2, ..., n\}$. In addition, for an element r_{ij} interpreting the preference degree of x_i over x_j, the following holds:

(i) if $r_{ij} = 0.5$, then it indicates indifference between x_i and x_j, or maximal fuzziness;

(ii) if $r_{ij} > 0.5$, then it indicates that x_i is preferred over x_j;

(iii) if $r_{ij} = 1$, it implies that alternative x_i is definitely preferred over alternative x_j (crisp case).

As usual, R satisfies the additive reciprocity property:

$$r_{ij} + r_{ji} = 1 \qquad i, j = 1, 2, ...n. \tag{5}$$

Example 3.2.1 *Using the example in Table 3, it is possible to present the preference matrices for each specialist, related to all craft beer styles.*

The corresponding preference matrices R_{F_1} (Maryan), R_{F_2} (Jonathan), R_{F_3} (Willian) and R_{F_4} (Caroline) related to all craft beer styles are described below.

$$\underset{R_{F1}}{\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.7 & 0.5 & 0.4 \\ 0.8 & 0.6 & 0.5 \end{pmatrix}} \quad \underset{R_{F2}}{\begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0.6 & 0.5 & 0.3 \\ 0.9 & 0.7 & 0.5 \end{pmatrix}} \quad \underset{R_{F3}}{\begin{pmatrix} 0.5 & 0.1 & 0.3 \\ 0.9 & 0.5 & 0.2 \\ 0.7 & 0.8 & 0.5 \end{pmatrix}} \quad \underset{R_{F4}}{\begin{pmatrix} 0.5 & 0.3 & 0.4 \\ 0.7 & 0.5 & 0.9 \\ 0.6 & 0.1 & 0.5 \end{pmatrix}}$$

One can notice that for R_{F_1} matrix, the sum of Maryan's preferences between the weiss and the sour styles has to be the value 1. For example, according to Eq.(5), as her preference value for the weiss style, over the sour, is 0.7 then the preference value for the sour style, over the weiss, is 0.3, meaning that, the sum of the preferences between these two styles is equal to 1. Analogously, this was also analyzed for all the other relations between styles.

As proposed in (BELIAKOV; CALVO; JAMES, 2014), consensus measures are defined by functions on the unit interval $[0, 1]$, modeling the agreement related to several inputs based on two main properties:

1. The unanimity, interpreting the complete consensus which is achieved when all inputs are the same;

2. The minimal consensus, which is related to the special case of two inputs, resulting in a null-consensus whenever one of theses inputs lies at one of the extremes (either 0, or 1) in the unit interval.

The main idea of a consensus measure is formalized as follows:

Definition 3.2.2 *(BELIAKOV; CALVO; JAMES, 2014, Definition 7). A function $C\colon [0, 1]^n \to [0, 1]$ is said to be a consensus measure if it satisfies the following properties:*

$\mathcal{C}1$: $C(a, a, ..., a) = 1$, $\forall a \in [0, 1]$ *(Unanimity);*

$\mathcal{C}2$: $C(0, 1) = C(1, 0) = 0$ *(Minimum consensus for n = 2).*

Considering further properties desired of consensus measures (BELIAKOV; CALVO; JAMES, 2014), we are focusing on the following:

$\mathcal{C}3$: $C(x_1, x_2, ..., x_n) = C(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)})$, for all σ-permutation on {1,...,n} and $(x_1, \ldots, x_n) \in [0, 1]^n$ (*Symmetry*);

$\mathcal{C}4$: $C(x_1, x_2, ..., x_n) = 0$, when $n = 2k$ and $k = \#\{x_i \colon x_i = 0\} = \#\{x_i \colon x_i = 1\}$ (*Maximum Dissension*);

$\mathcal{C}5$: $C(x_1, x_2, ..., x_n) = C(N(x_1), N(x_2), ..., N(x_n))$, when N is a strong fuzzy negation (*Reciprocity*);

$\mathcal{C}6$: $C(x) = C(x, x) = C(x, x, \ldots, x)$, $\forall \vec{x} \in [0, 1]^n$ (*Replication Invariance*);

$\mathcal{C}7$: For $n = 2k$, let half of the evaluations be equal, $\mathbf{a} = (a, a, ..., a)$ where $\mathbf{a} \in [0, 1]^k$. If $|a - x_j| \leq |a - y_j|$ for $j = 1, ..., k$, then $C(\mathbf{a}, x_1, x_2, ..., x_k) \geq C(\mathbf{a}, y_1, y_2, ..., y_k)$. (*Monotonicity w.r.t. the Majority*).

There are many works related to consensus measure which have been employed in decision making contexts. In most cases, fuzzy connectives and aggregation operators are considered, in order to relate the sets of membership degree which are obtained from expert opinions. Such measures have been applied in voting preferences aggregation (UNZU; VORSATZ, 2011) and used to inform consensus reaching processes (HERRERA-VIEDMA et al., 2007).

Consensus provides understanding by distinct ways in group decision making (GDM) contexts, briefly described below in accordance with (HERRERA-VIEDMA et al., 2007) and (ROTHSTEIN; BUTLER, 2006).

(i) The state of agreement in a group, meaning a common feeling between the individuals about the values in question. From this perspective, consensus has been denoted as a full and unanimous agreement, though it has been considered questionable, if such state is possible in real world and virtual situations.

(ii) Methodology to reach consensus, which is also related to the sense given above, implying in an evolution of the attest of the group for consensus with respect to their testimonies. And this evolution can be freed or facilitated by a specific individual.

(iii) Approach in which decisions should be meant in multi-person settings. It aims to achieve the consent, but not necessarily the agreement of the individuals, by arranging views of all parties involved to obtain a decision that will produce what will be useful to the entire group. It is not always related to a particular individual who may give consent to, not necessarily in his/her first choice, but because, for example, he/she wants to cooperate with the group.

The first approaches of consensus reaching process started in the late 1940s and early 1950s, with two main contributions, that are considered the beginning of participatory management in decision making, as shown in the review by (HERRERA-VIEDMA et al., 2014).

Later, the consensus theory is developed in a more general form in (SPILLMAN; SPILLMAN; BEZDEK, 1980). These initial formulations describe the formation of group consensus, but do not provide an adequate account of settled patterns of disagreement. Later, many models of consensus reaching (formation) have been proposed, notably in the domain of the so called rational consensus.

Then, in the year of 1985, the classical consensus approaches were introduced, where the notion of consensus has conventionally been understood in terms of strict and unanimous agreement.

Other important contributions in fuzzy consensus and fuzzy decision making appeared later in 1986 and 1996, as observed in the review done by in (HERRERA-VIEDMA et al., 2014).

It is possible to find different consensus approaches in the literature, according to the reference domain used to obtain the consensus measures:

A. Consensus measures focused on a set of experts, where consensus degrees are obtained in three steps:

(i) for each pair of individuals, is computed a degree of agreement as to their opinions between all the pairs of options,

(ii) these agreement degrees are combined to obtain a degree of agreement of each pair of individuals as to their preferences between pairs of options; and finally,

(iii) these agreement degrees are combined to obtain an agreement degree of pairs of individuals as to their preferences between pairs of options, which is the consensus degree of the group of experts.

B. Consensus measures can also focus on the alternative set, considering three levels of a preference relation:

(i) level of preference, which indicates the consensus degree existing among all the preference values attributed by the experts to a specific preference;

(ii) level of alternative, which allows us to measure the consensus existing over all the alternative pairs where a given alternative is present; and

(iii) level of preference relation, which evaluates the social consensus, that is, the current consensus existing among all the experts about all the preferences.

According to (HERRERA-VIEDMA et al., 2014), in order to guide the experts to change their preferences during the discussion process, the analysis of levels in preference relations seems to be adequate in consensus process designs. In addition, four current trends in the field of consensus models were also discussed:

(i) Adaptative consensus models, providing strategies adapting the number of changes in the GDM problem which are required to the experts in each round of consensus (MATA; MARTÍNEZ-LÓPEZ; HERRERA-VIEDMA, 2009);

(ii) Trust based consensus models, providing techniques to explore unsuitable specialists for the decision process, in order to consider a subgroup of relevant specialists improving the achievement of solutions of problems in GDM (ALONSO et al., 2013);

(iii) Dynamic and changeable consensus models, investigating new situations where alternatives might change and even disappear while experts are discussing and making decision (VICTOR et al., 2011);

(iv) Consensus models based on agent theory, as a tool to obtain alternatives based on anthologies providing an advanced representation of information for possible evaluation of alternatives analyzed by groups of specialists (PÉREZ; CABRERIZO; HERRERA-VIEDMA, 2011).

The use of these models is at an early stage of development and several future challenges still have to be solved.

3.2.1 Application of consensus measure algorithm

In (XIA; XU, 2011b), an algorithm is introduced, which can automatically modify the diverging individual fuzzy preference relations so as to reach an acceptable consensus. This prevents the specialists from changing their preferences and then, making the decision more efficient.

It is possible to describe a group decision making as follows: suppose that m decision makers $\mathbf{e}_l$ $(l = 1, 2, .., m)$ give their individual fuzzy preference relations $R_l = (r_{ijl})_{n\times n}$ $(l = 1, 2, .., m)$ over alternatives $x_1, x_2, ...x_n$, and $\lambda = (\lambda_1, \lambda_2, .., \lambda_m)^T$ is the weighting vector of the decision makers $\mathbf{e}_l$ $(l = 1, 2, .., m)$ with the condition that $\sum_{l=1}^{m} \lambda_l = 1$ and $0 \leq \lambda_l \leq 1,\ l = 1, 2, .., m$.

The following steps apply the algorithm proposed in (XIA; XU, 2011b, Alg. 4.1), using the idea presented in Table 3.

As presented in Example 3.2.1, we consider four specialists $\mathbf{e}_l$ $(l = 1, 2, 3, 4)$ providing ratings for three styles of craft beers, and then their corresponding matrices of fuzzy preference relations about three craft beer styles. So, in this example we applied the algorithm seen in (XIA; XU, 2011b, Alg. 4.1) to reach an acceptable group consensus, that is summarized in the following steps:

Step 1. It is constructed the multiplicative consistent fuzzy preference relations $\overline{R}_l$ $(l = 1, 2, 3, 4)$ from R_l $(l = 1, 2, 3, 4)$ using Eq. (6):

$$\overline{r}_{ik} = \frac{\sqrt[n]{\prod_{t=1}^{n}(r_{it}r_{tk})}}{\sqrt[n]{\prod_{t=1}^{n}(r_{it}r_{tk})} + \sqrt[n]{\prod_{t=1}^{n}((1 - r_{it})(1 - r_{tk}))}}\ ,\ i, k = 1, 2, .., n. \quad (6)$$

The matrices $\overline{R}_l$ of preference relation are reported in the following:

$$\overline{R}_1 = \begin{pmatrix} 0.5 & 0.709 & 0.793 \\ 0.291 & 0.5 & 0.611 \\ 0.207 & 0.389 & 0.5 \end{pmatrix} \qquad \overline{R}_3 = \begin{pmatrix} 0.5 & 0.783 & 0.853 \\ 0.217 & 0.5 & 0.616 \\ 0.147 & 0.384 & 0.5 \end{pmatrix}$$

$$\overline{R}_2 = \begin{pmatrix} 0.5 & 0.673 & 0.868 \\ 0.327 & 0.5 & 0.762 \\ 0.132 & 0.238 & 0.5 \end{pmatrix} \qquad \overline{R}_4 = \begin{pmatrix} 0.5 & 0.807 & 0.455 \\ 0.193 & 0.5 & 0.166 \\ 0.545 & 0.834 & 0.5 \end{pmatrix}$$

Step 2. Then, the individual fuzzy preference relations $\overline{R}_l$ $(l = 1, 2, 3, 4)$ are aggregated into a group fuzzy preference relation $\overline{R}$ according to Eq. (7). For convenience, let $s = 0$, $\overline{R}_l^{(0)} = (\overline{r}_{ijl}^{(0)})_{n\times n} = \overline{R}_l = (\overline{r}_{ijl})_{n\times n}$, $\overline{R}^{(0)} = (\overline{r}_{ij}^{(0)})_{n\times n} = \overline{R} = (\overline{r}_{ij})_{n\times n}$, $l = 1, 2, 3, 4$, and

$$\overline{r}_{ij} = \frac{\prod_{l=1}^{m}(r_{ijl})^{\lambda_l}}{\prod_{l=1}^{m}(r_{ijl})^{\lambda_l} + \prod_{l=1}^{m}(1 - r_{ijl})^{\lambda_l}}\ ,\ i, j = 1, 2, ..n, \quad (7)$$

then, we obtain the matrice $\overline{R}^{(0)}$ as follows:

$$\overline{R}^{(0)} = \begin{pmatrix} 0.5 & 0.747 & 0.769 \\ 0.253 & 0.5 & 0.530 \\ 0.231 & 0.470 & 0.5 \end{pmatrix}$$

Step 3. The deviation degree between each individual preference relation $\overline{R}_l^{(0)}$ and the group preference relation $\overline{R}^{(0)}$ is calculated by Eq. (8):

$$d(\overline{R}_l^{(s)}, \overline{R}^{(s)}) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}|r_{ijl}^{(s)} - r_{ij}^{(s)}|. \tag{8}$$

The result parameters are reported in the sequence:

$$d(\overline{R}^{(0)}, \overline{R}_1^{(0)}) = 0.0317 \qquad d(\overline{R}^{(0)}, \overline{R}_2^{(0)}) = 0.0901$$
$$d(\overline{R}^{(0)}, \overline{R}_3^{(0)}) = 0.0461 \qquad d(\overline{R}^{(0)}, \overline{R}_4^{(0)}) = 0.1638$$

As the algorithm indicates, without loss of generality, in this example it is considered $\rho = 0.05$. Thus,

$$d(\overline{R}^{(0)}, \overline{R}_2^{(0)}) = 0.0901 \ > \ 0.05 \quad \text{and} \quad d(\overline{R}^{(0)}, \overline{R}_4^{(0)}) = 0.1638 \ > \ 0.05,$$

and then, it is necessary to perform Step 4.

Step 4. Supposing a normalization $\eta = 0.5$, then it is calculated again the terms $\overline{R}_l^{(1)} = (\overline{r}_{ijl}^{(1)})_{n\times n}$, $l = 2, 4$ and $\overline{R}^{(1)} = (\overline{r}_{ij}^{(1)})_{n\times n}$, where $\overline{R}_l^{(1)} = \overline{R}_l^{(0)},\ l = 1, 3$ and

$$\overline{r}_{ijl}^{(1)} = \frac{(\overline{r}_{ijl}^{(0)})^{(1-\eta)}(\overline{r}_{ij}^{(0)})^{\eta}}{(\overline{r}_{ijl}^{(0)})^{(1-\eta)}(\overline{r}_{ij}^{(0)})^{\eta} + (1-\overline{r}_{ijl}^{(0)})^{(1-\eta)}(1-\overline{r}_{ij}^{(0)})^{\eta}}\ ,\ i, j, l = 2, 4, \tag{9}$$

$$r_{ij}^{(1)} = \frac{\prod_{l=1}^{m}(r_{ijl}^{(1)})^{\lambda_l}}{\prod_{l=1}^{m}(r_{ijl}^{(1)})^{\lambda_l} + \prod_{l=1}^{m}(1-r_{ijl}^{(1)})^{\lambda_l}}\ ,\ i, j = 1, 2, 3, 4. \tag{10}$$

Thus:

$$\overline{R}_2^{(1)} = \begin{pmatrix} 0.5 & 0.711 & 0.824 \\ 0.289 & 0.5 & 0.655 \\ 0.176 & 0.345 & 0.5 \end{pmatrix} \qquad \overline{R}_4^{(1)} = \begin{pmatrix} 0.5 & 0.779 & 0.625 \\ 0.221 & 0.5 & 0.322 \\ 0.375 & 0.678 & 0.5 \end{pmatrix}$$

$$\overline{R}^{(1)} = \begin{pmatrix} 0.5 & 0.747 & 0.784 \\ 0.253 & 0.5 & 0.551 \\ 0.216 & 0.449 & 0.5 \end{pmatrix}$$

So, let $s = 1$, and return to Step 3, once it is necessary to recalculate the deviation degree between each individual preference relation $\overline{R}_l^{(1)}$ and the group preference relation $\overline{R}^{(1)}$ by Eq. (8):

$$d(\overline{R}^{(1)}, \overline{R}_1^{(1)}) = 0.0317 \qquad d(\overline{R}^{(1)}, \overline{R}_2^{(1)}) = 0.0480$$
$$d(\overline{R}^{(1)}, \overline{R}_3^{(1)}) = 0.0461 \qquad d(\overline{R}^{(1)}, \overline{R}_4^{(1)}) = 0.0852$$

Since $d(\overline{R}^{(1)}, \overline{R}_4^{(1)}) = 0.0852 > 0.05$, it is necessary to go back to Step 4, obtaining $\overline{R}_1^{(2)} = \overline{R}_1^{(1)}$, $\overline{R}_2^{(2)} = \overline{R}_2^{(1)}$, $\overline{R}_3^{(2)} = \overline{R}_3^{(1)}$ and

$$\overline{R}_4^{(2)} = \begin{pmatrix} 0.5 & 0.763 & 0.711 \\ 0.237 & 0.5 & 0.433 \\ 0.289 & 0.567 & 0.5 \end{pmatrix} \qquad \overline{R}^{(2)} = \begin{pmatrix} 0.5 & 0.743 & 0.800 \\ 0.257 & 0.5 & 0.580 \\ 0.200 & 0.420 & 0.5 \end{pmatrix}$$

So, by Eq. (8), we reach the following values:

$$d(\overline{R}^{(2)}, \overline{R}_1^{(2)}) = 0.0317 \qquad d(\overline{R}^{(2)}, \overline{R}_2^{(2)}) = 0.0480$$
$$d(\overline{R}^{(2)}, \overline{R}_3^{(2)}) = 0.0461 \qquad d(\overline{R}^{(2)}, \overline{R}_4^{(2)}) = 0.0380$$

Finally, all of the deviation relations are less then 0.05, thus the acceptable consensus of the group is achieved.

3.3 Chapter summary

In this chapter, we reported the concepts on fuzzy set theory, revisiting the basic concepts of aggregation functions on the unit interval $[0, 1]$. Some properties and exemplifications were presented, including an important class of triangular norms. Besides, the main properties of fuzzy connectives were reported, such as fuzzy negations and implication functions. Extended fuzzy aggregation operators and partial orders on fuzzy set theory were also considered in this chapter, as well as the notion of fuzzy preference relation and after, the axiomatic definition of fuzzy consensus measures. The chapter concludes with an exemplification using a consensus measure algorithm.

4 THEORY OF TYPICAL HESITANT FUZZY SETS

Hesitant Fuzzy Sets (HFS) were introduced by Torra in (TORRA; NARUKAWA, 2009) and Torra and Narukawa in (TORRA, 2010). In their work, the membership degree of an element that belongs to a set was represented by means of a subset of $[0, 1]$.

In the process of decision-making, HFS can be useful to handle situations where there is indecision among many possible values for the preferences over objects.

In this chapter, the notion of some typical hesitant fuzzy connectives is presented, based on an admissible partial order $\leq$ related to the poset $(\mathbb{H}, \leq)$. Then, is presented three new admissible orders in the typical hesitant fuzzy setting, which will allow us to introduce the notion of some typical hesitant connectives.

4.1 Typical hesitant fuzzy sets

Formally, let $\wp([0, 1])$ be the power set of $[0, 1]$. A HFS A defined over U, where U is a non-empty set, is given by:

$$A = \{(x, \mu_A(x)) : x \in U\}, \tag{11}$$

where $\mu_A : U \to \wp([0, 1])$ is the membership function. In particular, when $\mu_A(x)$ is finite and non-empty for each $x \in U$, in this case we have a Typical Hesitant Fuzzy Set (THFS).

Definition 4.1.1 *(BEDREGAL et al., 2014a) Let $\mathbb{H} = \{X \subseteq [0, 1] : X$ is finite and $X \neq \emptyset\}$. A THFS A defined over U is given by Eq. (11), where $\mu_A : U \to \mathbb{H}$.*

Each $X \in \mathbb{H}$ is named Typical Hesitant Fuzzy Element (THFE) of $\mathbb{H}$ and the cardinality of X, i.e. the number of elements of X, is referred to as $\#X$. The i^{th} smallest element of a THFE X will be denoted by $X^{(i)}$.

The bottom and the top elements $\{0\}$ and $\{1\}$ will be denoted, respectively, by $\mathbf{0}_{\mathbb{H}}$ and $\mathbf{1}_{\mathbb{H}}$. The set of all unitary subsets on $\wp([0, 1])$ is called the set of diagonal or degenerate elements of $\mathbb{H}$ and will be denoted by $\mathcal{D}_{\mathbb{H}}$, i.e. $\mathcal{D}_{\mathbb{H}} = \{X \in \mathbb{H} \colon \#X = 1\}$.

Some examples of THFS are $X = \{0.3, 0.4, 0.7\}$ and $Y = \{0.5, 0.6\}$ where $\#X = 3$ and $\#Y = 2$. In these examples, $X^{(1)} = 0.3$ and $Y^{(2)} = 0.6$.

In (BEDREGAL et al., 2014, Definition 3.2), a function $\mathcal{F} : \mathbb{H}^n \to \mathbb{H}$ is said to preserve degenerate elements if $\mathcal{F}(\mathcal{D}_{\mathbb{H}}) \subseteq \mathcal{D}_{\mathbb{H}}$, i.e. if it satisfies the property:

$\mathbf{DP}$: $\mathcal{F}(\{x_1\}, \ldots, \{x_n\}) \in \mathcal{D}_{\mathbb{H}}$ for each $x_1, \ldots, x_n \in [0, 1]$.

According with (BEDREGAL et al., 2014, Example 3.1), if $\mathcal{F} \colon \mathbb{H}^n \to \mathbb{H}$ preserves degenerate elements then, the function $F : [0, 1]^n \to [0, 1]$ such that $\forall x_1, \ldots, x_n \in [0, 1]$:

$$\{F(x_1, \ldots, x_n)\} = \mathcal{F}(\{x_1\}, \ldots, \{x_n\}) \tag{12}$$

is well defined and it is said to be induced from $\mathcal{F}$

In addition, based on (BEDREGAL et al., 2014, Example 3.2), any function F : $[0, 1]^n \to [0, 1]$ induces a mapping $\mathcal{F} : \mathbb{H}^n \to \mathbb{H}$ which preserves degenerate elements as follows:

$$\mathcal{F}(X_1, \ldots, X_n) = \{F(x_1, \ldots, x_n) : x_i \in X_i, \forall i \in \mathcal{N}_n\}, \forall X_1, \ldots, X_n \in \mathbb{H}, \tag{13}$$

where $\mathcal{N}_n = \{1, \ldots, n\}$.

4.2 $\langle \mathbb{H}, \leq \rangle$-partial orders

There are several proposals of orders for THFE, as for example the ones given in (BEDREGAL et al., 2014a; WANG; XU, 2016; XU; XIA, 2011; ZHANG; YANG, 2016, 2015). The unique consensus among all these orders is that all of them refines[1] the following restrictive partial $\langle \mathbb{H}, \leq_R H \rangle$-order:

$$X \leq_{RH} Y \text{ iff } X = \mathbf{0}_{\mathbb{H}} \text{ or } Y = \mathbf{1}_{\mathbb{H}} \text{ or } (\#X = \#Y = n \text{ and } X^{(i)} \leq Y^{(i)}, \forall i \in \mathbb{N}_n). \tag{14}$$

Since, trivially, for each $X \in \mathbb{H}$, we have that $\mathbf{0}_{\mathbb{H}} \leq_{RH} X \leq_{RH} \mathbf{1}_{\mathbb{H}}$, then $\mathbf{0}_{\mathbb{H}}$ is the bottom and $\mathbf{1}_{\mathbb{H}}$ is the top element of the poset $\langle \mathbb{H}, \leq_{RH} \rangle$.

In a restrictive order approach, the condition enabling a comparison between two THFE $X, Y \in \mathbb{H}$, which are different from $\mathbf{0}_{\mathbb{H}}$ and $\mathbf{1}_{\mathbb{H}}$, is that they have the same cardinality, meaning that $\#X = \#Y$.

Example 4.2.1 *Take the partial order introduced by Xu and Xia (XU; XIA, 2011), that for two HFE with different cardinalities, it is added the minimum element (or maximum element) to the shortest HFE up to both of the HFE has the same cardinality. In (SANTOS et al., 2014; XU; LIU; ZHANG, 2019), formal definitions were given for Xu*

[1] A partial $\leq_1$-order on a set S refines another partial order $\leq_2$ on S if $(S, \leq_2) \subseteq (S, \leq_1)$, i.e. for each $x, y \in S$ such that $x \leq_2 y$ we have that $x \leq_1 y$.

and Xia's order ($\leq_{XX}$) transforming elements of $\mathbb{H}$ in elements of other sets. In the following, we provide a direct formal definition of $\leq_{XX}$, i.e. not using the aforementioned approach. Let $X, Y \in \mathbb{H}$, then

$$X \leq_{XX} Y \text{ iff } \begin{cases} X^{\underline{n}} \leq_{RH} Y \text{ and } n \leq m; \text{ or} \\ X \leq_{RH} Y^{\underline{m}} \text{ and } X^{(1)} \leq Y^{(1)} \text{ and } m \leq n, \end{cases} \tag{15}$$

where $m = \#X$, $n = \#Y$ and $Z^{\underline{k}} = \{Z^{(\#Z-i+1)} : i \in \mathbb{N}_k\}$ for $k \leq \#Z$.

By preserving the same cardinality of HFS based on repetitions of THFE, another way to generate partial orders for THFE is considering the following strategy.

Example 4.2.2 *In (FARHADINIA, 2016), given in Definition 3.2, which is defined for $X, Y \in \mathbb{H}^{(n)}$ by requiring*

$$X <_{Lex} Y \Leftrightarrow \exists i, 1 \leq i \leq n \text{ such that } X^{(j)} = Y^{(j)} \text{ for } j < i \text{ and } X^{(i)} < Y^{(i)}.$$

Based on the above lexicographical ordering, Farhadinia presents a "novel HFS ranking technique" in Definition 3.3 of (FARHADINIA, 2016), the ranking vector is a pair-composed function given by the score function S_{AM} (as the arithmetic mean) and the successive deviation function ϑ, defined as follows:

$$R(X) = (S_{AM}(X), \vartheta_2(X)) = \left(\frac{1}{n} \sum_{i=1}^{n} X^{(i)} \ , \ \sum_{i=1}^{n-1} \left(X^{(i+1)} - X^{(i)} \right)^2 \right).$$

For $X, Y \in \mathbb{H}^{(n)}$, the $\langle \mathbb{H}^{(n)}, \leq_R \rangle$-relation is established as follows:

$$X <_R Y \Leftrightarrow R(X) <_{Lex} R(Y); X \leq_R Y \Leftrightarrow R(X) \leq_{Lex} R(Y); \text{ and } X = Y \Leftrightarrow R(X) = R(Y);$$

where for each $(a,b), (c,d) \in [0,1]^2$, $(a,b) \leq_{Lex} (c,d)$ iff $a < c$ or ($a = c$ and $b \leq d$). Nevertheless, $\leq_R$ is not an order on $\mathbb{H}^{(n)}$ since the antisymmetry fails. Moreover, such relation is not an order on $\mathbb{H}$ and therefore can not be considered as HFS ranking technique. In fact, consider $X = \{0, 0.3, 0.6, 0.7\}$ and $Y = \{0.1, 0.2, 0.5, 0.8\}$. Then $S_{AM}(X) = 0.4 = S_{AM}(Y)$ and $\vartheta_2(X) = 0.19 = \vartheta_2(Y)$, i.e. $X \leq_R Y$ and $Y \leq_R X$.

Example 4.2.3 *Let $\preceq_{Lex}$ be the order on $\mathbb{H}^{(n)}$ proposed in (WANG; XU, 2016, Example 2), which is given as follows:*

$$X \preceq_{Lex} Y \Leftrightarrow \begin{cases} X = Y \text{ or} \\ \exists m > 0, \text{ such that } \forall i < m, X^{(i)} = Y^{(i)} \text{ and } X^{(m)} < Y^{(m)}. \end{cases} \tag{16}$$

Thus, $\preceq_{Lex}$ is a very restrictive order, because it only enables the comparison between two THFE with the same cardinality, i.e. this relation is a linear order on $\mathbb{H}^{(n)}$. There is an extension of such order in (BEDREGAL et al., 2014a, Remark 1) for $\mathbb{H}$, i.e. it allows to compare THFE of different cardinalities, but this order is not linear.

Our aim in the present work is to establish an admissible order to allow comparisons between THFE without this restriction. The idea of admissible order was presented in (BUSTINCE et al., 2013) for interval-valued fuzzy sets and after in (MIGUEL et al., 2016) for interval-valued Atanassov's intuitionistic fuzzy sets. We acknowledge that some efforts have already been made in order to establish an admissible ordering for hesitant fuzzy sets, as seen in (WANG; XU, 2016). However, their proposal requires that both THFE must have the same cardinality.

In the next section, we present an admissible order in the typical hesitant fuzzy setting, which will allow us to introduce the notion of some typical hesitant connectives.

4.3 Admissible $\langle\mathbb{H}, \preceq\rangle$-orders for typical hesitant fuzzy sets

In order to allow to overcome the main restrictions of typical hesitant fuzzy sets, that is, the difficulty to consolidate methods to enable the ordering of typical hesitant fuzzy elements and comparisons, this chapter presents a new Admissible $\mathbb{H}$ Order for the HFS elements and also allow us to introduce some operators, such as the typical hesitant fuzzy negations and typical hesitant t-norms introduced in the next chapter.

4.3.1 Admissible $\mathbb{H}$ orders for typical hesitant fuzzy elements

Although the principal order relation we use in $\mathbb{H}$ is the standard partial order $\leq_{RH}$, the existence of many incomparable pairs of THFE is an undesirable situation in some applications as well as in the study of important properties of typical hesitant fuzzy connectives, mainly related to implications, negations and aggregation functions. Thus, we make use of admissible $\langle\mathbb{H}, \preceq\rangle$-orders, a family of total orders that refine the partial $\langle\mathbb{H}, \leq_{RH}\rangle$-order.

Definition 4.3.1 *A partial $\langle\mathbb{H}, \preceq\rangle$-order is admissible if it refines $\langle\mathbb{H}, \leq_{RH}\rangle$.*[2]

From now on, admissible total orders will be simply referred to as admissible orders. And, the expressions $\langle\mathbb{H}, \leq\rangle$ and $\langle\mathbb{H}, \preceq\rangle$ denote the poset of all THFE w.r.t. the partial $\leq$-order and to the admissible (total) $\preceq$-order on $\mathbb{H}$, respectively. Moreover, $\mathfrak{O}_{\mathfrak{P}}$ and $\mathfrak{O}_{\mathfrak{A}}$ denote the sets of all partial orders and all admissible orders on $\mathbb{H}$, respectively. In addition, $\langle\mathbb{H}, \preceq, \mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}\rangle$ is a chain and therefore a complete lattice. See (MATZENAUER

[2]Note that the admissible orders in Definition 4.3.1 is not equivalent to the notion of admissible order for hesitant fuzzy elements (HFE) presented in (WANG; XU, 2016). In fact, it proposes a total order for HFE restricted to a size n which refines the order $\leq_{RH}$ restricted to $\mathbb{H}^{(n)}$.

et al., 2020), as a first attempt of defining an admissible order on $\mathbb{H}$ in a more restrictive way, moving to a more general order relation as follows.

4.3.1.1 Admissible $\langle \mathbb{H}, \preceq_{Lex1} \rangle$ and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$ orders for typical hesitant fuzzy elements

In this subsection, we introduce the concepts of $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders for typical hesitant fuzzy elements, which are admissible total orders, i.e. both are totals and refines $\langle \mathbb{H}, \leq_{RH} \rangle$.

Theorem 4.3.1 *The relations $\preceq_{Lex1}$ and $\preceq_{Lex2}$ on $\mathbb{H}$, are defined, respectively, as:*

$$X \preceq_{Lex1} Y \Leftrightarrow \begin{cases} \exists i \in \mathbb{N}_{\min(m,n)} \colon X^{(i)} < Y^{(i)} \text{ and } X^{(j)} = Y^{(j)}, \forall j < i; \text{ or} \\ m \leq n \text{ and } X^{(j)} = Y^{(j)}, \forall j \in \mathbb{N}_m, \end{cases} \tag{17}$$

$$X \preceq_{Lex2} Y \Leftrightarrow \begin{cases} \exists i \in \mathbb{N}_{\min(m,n)} \colon X^{(\#X-i+1)} < Y^{(\#Y-i+1)} \text{ and} \\ X^{(\#X-i+j+1)} = Y^{(\#Y-i+j+1)}, \forall j < i; \qquad \text{or} \\ m \leq n \text{ and } X^{(\#X-j+1)} = Y^{(\#X-j+1)}, \forall j \in \mathbb{N}_m; \end{cases} \tag{18}$$

where $m = \#X$ and $n = \#Y$, are admissible total orders, i.e. $\preceq_{Lex1}, \preceq_{Lex2} \in \mathfrak{O}_{\mathfrak{A}}$.

Proof: The proof related to $\preceq_{Lex1}$-order is presented and the other one can be analogously done. Trivially $\preceq_{Lex1}$ is reflexive and in case $X \preceq_{Lex1} Y$ and $Y \preceq_{Lex1} X$ for some $X, Y \in \mathbb{H}$, then by Eq. (17), we have the following cases:

(i) Suppose that $X \neq Y$. Then, (a) $\exists i \in \mathbb{N}_{\min(m,n)} \colon X^{(i)} < Y^{(i)}$ and $X^{(j)} = Y^{(j)}, \forall j < i$; and (b) $\exists k \in \mathbb{N}_{\min(m,n)} \colon Y^{(k)} < X^{(k)}$ and $X^{(j)} = Y^{(j)}, \forall j < k$. If $i < k$, then by (b) we have that $X^{(i)} = Y^{(i)}$ which is a contradiction with (a). Analogously, if $k < i$, by (a) and (b) we will have that $X^{(k)} = Y^{(k)}$ and $Y^{(k)} < X^{(k)}$ which is also a contradiction. Finally, in case $i = k$, from (a) and (b) we would have $X^{(i)} < Y^{(i)}$ and $Y^{(i)} < X^{(i)}$ which is also a contradiction.

(ii) (a) $m \leq n$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Y^{(j)}$ and (b) $\exists i \in \mathbb{N}_m \colon Y^{(i)} < X^{(i)}$ and $X^{(j)} = Y^{(j)}, \forall j < i$. So, by (a) $X^{(i)} = Y^{(i)}$ and by (b) $Y^{(i)} < X^{(i)}$ which is a contradiction.

(iii) (a) $\exists i \in \mathbb{N}_{\min(m,n)} \colon X^{(i)} < Y^{(i)}$ and $X^{(j)} = Y^{(j)}, \forall j < i$, and (b) $n \leq m$ and $\forall j \in \mathbb{N}_n, X^{(j)} = Y^{(j)}$. So, by (a) $X^{(i)} < Y^{(i)}$ for some $i \in \mathbb{N}_n$. By (b), $X^{(i)} = Y^{(i)}$ which is a contradiction.

Therefore, the unique possibility is that: (a) $m \leq n$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Y^{(j)}$ and (b) $n \leq m$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Y^{(j)}$. But in this case $m = n$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Y^{(j)}$ which means that $X = Y$. Hence $\preceq_{Lex1}$ is antisymmetric.
Finally, suppose that $X \preceq_{Lex1} Y \preceq_{Lex1} Z$ for some $X, Y, Z \in \mathbb{H}$. In case $X = Y$ or

$Y = Z$, trivially $X \preceq_{Lex1} Z$. Let $m = \#X$, $n = \#Y$ and $p = \#Z$. So, we have the following possibilities:

(i) (a) $\exists i \in \mathbb{N}_{\min(m,n)} \colon X^{(i)} < Y^{(i)}$ and $X^{(j)} = Y^{(j)}, \forall j < i$; and
(b) $\exists k \in \mathbb{N}_{\min(n,p)} \colon Y^{(k)} < Z^{(k)}$ and $Y^{(j)} = Z^{(j)}, \forall j < k$. If $i < k$, then by (b) we have that $Y^{(j)} = Z^{(j)}$ and for all $j \leq i$ therefore by (a), we have that $\exists i \in \mathbb{N}_{\min(m,p)} \colon X^{(i)} < Y^{(i)}$ and $X^{(j)} = Y^{(j)}, \forall j < i$, i.e. $X \prec_{Lex1} Z$. Analogously, if $k < i$, by (a) and (b) we will have that $X^{(k)} = Y^{(k)}$ and $Y^{(k)} = Z^{(k)}$ and for each $j < i$, $X^{(k)} = Y^{(k)} = Z^{(k)}$. So, $X \prec_{Lex1} Z$. Finally, in case $i = k$, then from (a) and (b) we would have $X^{(i)} < Y^{(i)}$ and $Y^{(i)} < Z^{(i)}$ and for each $j < i$, $X^{(k)} = Y^{(k)} = Z^{(k)}$. So, $X \prec_{Lex1} Z$.

(ii) (a) $m < n$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Y^{(j)}$; and
(b) $\exists i \in \mathbb{N}_{\min(n,p)} \colon Y^{(i)} < Z^{(i)}$ and $Y^{(j)} = Z^{(j)}, \forall j < i$. If $\min(m,p) = p$, then from (a) and (b), $X^{(i)} < Z^{(i)}$ and $X^{(j)} = Z^{(j)}, \forall j < i$. So, $X \prec_{Lex1} Z$. Else, if $\min(m,p) = m$ and $i > m$, then by (a) and (b), we have that $m \leq p$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Z^{(j)}$ and therefore $X \preceq_{Lex1} Z$. Finally, in case $\min(m,p) = m$ and $i \leq m$ then by (a) and (b), we have that $X^{(i)} < Z^{(i)}$ and $X^{(j)} = Z^{(j)}, \forall j < i$, i.e. $X \prec_{Lex1} Z$.

(iii) (a) $\exists i \in \mathbb{N}_{\min(m,n)} \colon X^{(i)} < Y^{(i)}$ and $X^{(j)} = Y^{(j)}, \forall j < i$; and
(b) $n < p$ and $\forall j \in \mathbb{N}_n, X^{(j)} = Y^{(j)}$. If $\min(m,n) = m$, then $m < p$ and from (a) and (b), $X^{(i)} < Y^{(i)} = Z^{(i)}$ and $X^{(j)} = Y^{(j)} = Z^{(j)}, \forall j < i \leq n$, i.e. $X \leq_{Lex1} Z$. If $\min(m,n) = n$, then from (a) and (b), $X^{(i)} < Y^{(i)} = Z^{(i)}$ and $X^{(j)} = Y^{(j)} = Z^{(j)}, \forall j < i$, i.e. $X \preceq_{Lex1} Z$.

(iv) (a) $m < n$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Y^{(j)}$; and (b) $n < p$ and $\forall j \in \mathbb{N}_n, Y^{(j)} = Z^{(j)}$. So, $m < p$ and $\forall j \in \mathbb{N}_m, X^{(j)} = Y^{(j)} = Z^{(j)}$ which means that $X \preceq_{Lex1} Z$.

So, $\preceq_{Lex1}$ is transitive and, therefore, it is a partial order which is trivially total and refines the partial $\langle \mathbb{H}, \leq_{RH} \rangle$-order. Therefore, Theorem 4.3.1 is verified. □

4.3.1.2 Admissible $\langle \mathbb{H}, \preceq_A^f \rangle$ orders

The admissible $\langle \mathbb{H}, \preceq_A^f \rangle$-orders is a family of total orders that refine the partial $\langle \mathbb{H}, \leq_{RH} \rangle$-order. In this section, we present the concept of $\langle \mathbb{H}, \preceq_A^f \rangle$-orders for typical hesitant fuzzy elements.

Theorem 4.3.2 *Let $A^* : \mathbb{H} \to [0,1]$ be a function such that A^* is increasing w.r.t. $\leq_{RH}$, $A^*(\mathbf{0}_{\mathbb{H}}) = 0$ and $A^*(\mathbf{1}_{\mathbb{H}}) = 1$ and $f^* : \mathbb{H} \to \mathbb{R}$ be a function such that the property:*

$IC : f^*(X) = f^*(Y) \Rightarrow \#X = \#Y$ *(injective-cardinality property)* *is satisfied. The relation $\preceq^{f^*}_{A^*}$ on $\mathbb{H}$ defined by*

$$X \preceq^{f^*}_{A*} Y \Leftrightarrow \begin{cases} X = Y, \text{ or} \\ A^*(X) < A^*(Y), \text{ or} \\ A^*(X) = A^*(Y) \text{ and } f^*(X) < f^*(Y), \end{cases} \tag{19}$$

is a total admissible order on $\mathbb{H}$ whenever, for each $n \in \mathbb{N}^+$, $A^*_n = A^* \restriction \mathbb{H}^{(n)}$ is injective.

Proof: Trivially, the $\preceq^{f^*}_{A^*}$-order is reflexive and antisymmetric. Let $X, Y, Z \in \mathbb{H}$. If $X \preceq^{f^*}_{A^*} Y$ and $Y \preceq^{f^*}_{A^*} Z$ then, in case $X = Y$ or $Y = Z$, we immediately have that $X \preceq^{f^*}_{A^*} Z$. If $X \preceq^{f^*}_{A^*} Y$ and $Y \preceq^{f^*}_{A^*} Z$, the next cases hold:

1. $A^*(X) < A^*(Y)$ and $A^*(Y) < A^*(Z)$, then $A^*(X) < A^*(Z)$;
2. $A^*(X) < A^*(Y)$, $A^*(Y) = A^*(Z)$ and $f^*(Y) < f^*(Z)$, which means that $A^*(X) < A^*(Z)$;
3. $A^*(X) = A^*(Y)$, $f^*(X) < f^*(Y)$ and also consider that $A^*(Y) < A^*(Z)$, then the following in equation is verified: $A^*(X) < A^*(Z)$;
4. $A^*(X) = A^*(Y)$ and $f^*(X) < f^*(Y)$. When we also consider that $A^*(Y) = A^*(Z)$ and $f^*(Y) < f^*(Z)$. Then, we have that $A^*(X) = A^*(Z)$ and $f^*(X) < f^*(Z)$.

All of the above cases imply that $X \prec^{f^*}_{A^*} Z$. So, we have a transitive $\langle \mathbb{H}, \preceq^{f^*}_{A^*} \rangle$-relation. In addition, for each $X, Y \in \mathbb{H}$ such that $X \neq Y$, we have the following three cases:

1. $A^*(X) < A^*(Y)$ and so, it implies that $X \prec^{f^*}_{A^*} Y$;
2. $A^*(Y) < A^*(X)$ and so, it implies that $Y \prec^{f^*}_{A^*} X$;
3. $A^*(X) = A^*(Y)$ so, if $f^*(X) < f^*(Y)$, then $X \prec^{f^*}_{A^*} Y$ and if $f^*(Y) > f^*(X)$, then $Y \prec^{f^*}_{A^*} X$. Finally, if $f^*(X) = f^*(Y)$, then because f^* satisfies the property IC, we have that $\#X = \#Y$ and so, since $A^*_{\#X}$ is injective, then $X = Y$, which is a contradiction.

Therefore, we have that $\langle \mathbb{H}, \preceq^{f^*}_{A^*} \rangle$-order is total. Finally, let $X, Y \in \mathbb{H}$ such that $X \leq_{RH} Y$. If $X = Y$, then by the reflexivity property, $X \preceq^{f^*}_{A^*} Y$ and if $X <_{RH} Y$, then we have three cases:

1. $X = \mathbf{0}_{\mathbb{H}}$ and $Y \neq \mathbf{0}_{\mathbb{H}}$, and in this case $A^*(\mathbf{0}_{\mathbb{H}}) = 0$ and $A^*(Y) \neq 0$ because $A^*_{\#Y}$ is injective and $A^*_{\#Y}(0, \ldots, 0) = 0$. Therefore, $A^*(X) < A^*(Y)$ which means that $X \prec^{f^*}_{A^*} Y$;
2. $X \neq \mathbf{1}_{\mathbb{H}}$ and $Y = \mathbf{1}_{\mathbb{H}}$, which is analogous to the previous case;

3. Neither $X = \mathbf{0}_{\mathbb{H}}$, nor $Y = \mathbf{1}_{\mathbb{H}}$, and in this case $\#X = \#Y = n$ and $X^{(i)} \leq Y^{(i)}$ for each $i \in \mathbb{N}_n$. So, $A^*_n(X) < A^*_n(Y)$ since A^*_n is injective and increasing. Therefore, $X \prec^{f^*}_{A^*} Y$.

Therefore, one can conclude that $\langle \mathbb{H}, \preceq^{f^*}_{A^*} \rangle$-order is admissible and total. □

Observe that A^* can be seen as an EAF. In general, it is not easy to find an EAF such that the family of aggregation functions be injective. In fact, for each $n > 1$ and an n-ary aggregation function A^*_n, if A^*_n is either idempotent or has a neutral or an annihilator element, then A^*_n is not injective. Moreover, if A^*_n is injective then it is neither conjunctive, nor disjunctive nor average aggregation function (BELIAKOV; BUSTINCE; CALVO, 2016) meaning that A^*_n is a mixed aggregation function. Moreover, it is clear that not all mixed aggregation functions are injective (e.g. proper uninorms and proper nullnorms). See more details in the literature (BELIAKOV; PRADERA; CALVO, 2007).

Given $x \in [0,1]$ and $i \in \mathbb{N}^+$, denote by $x^{[i]}$ the i-th decimal digit of the proper decimal expansion[3] of x, in case $x \neq 1$, and 9 in case $x = 1$. Since $x \in [0,1]$ and $1 = 0.\overline{9}$, the integer part of x, i.e. $x^{[0]}$, will always be 0. Thus, if $x = \pi - 3$, $y = 0.25$ and $z = 1$ then we have that: (i) $x^{[0]} = 0$, $x^{[1]} = 1$, $x^{[2]} = 4$, $x^{[3]} = 1$, $x^{[4]} = 5$, etc.; (ii) $y^{[0]} = 0$, $y^{[1]} = 2$, $y^{[2]} = 5$ and $y^{[i]} = 0$, for all $i \in \mathbb{N}^{+2} = \{3, 4, \ldots\}$; (iii) $z^{[0]} = 0$ and $z^{[i]} = 9$, for each $i \in \mathbb{N}^+$.

Inspired in a function proposed in (SANTANA et al., 2020), where the decimal expansion of two ordered values in $[0,1]$ are mixed, we introduce a function which mixes the (ordered) values of THFE, as presented in the next proposition.

Proposition 4.3.1 *Let $A : \mathbb{H} \to [0,1]$ be the function such that, for each $i \in \mathbb{N}^+$, $A(X)^{[i]} = \left(X^{(k)}\right)^{[j]}$ for $k = ((i-1) \mod \#X) + 1$ and $j = \left\lceil \frac{i}{\#X} \right\rceil$. Then, A satisfies the property of Theorem 4.3.2. And, $\preceq^f_A$ is a total admissible order on $\mathbb{H}$, if $f^* : \mathbb{H} \to \mathbb{R}$ satisfies IC.*

Proof: Trivially $A(\mathbf{0}_{\mathbb{H}}) = 0$ and $A(\mathbf{1}_{\mathbb{H}}) = 1$. Now, if $X \leq_{RH} Y$ then we have four cases.

1. If $X = \mathbf{0}_{\mathbb{H}}$, then $A(X) = 0 \leq A(Y)$;

2. If $Y = \mathbf{1}_{\mathbb{H}}$, then $A(X) \leq 1 = A(Y)$;

3. If $\#X = \#Y$, $X \neq Y$ and $X^{(k)} \leq Y^{(k)}$ for each $k \in \mathbb{N}_{\#X}$, then there exists $j_k \in \mathbb{N}$, $\left(X^{(k)}\right)^{[j]} < \left(Y^{(k)}\right)^{[j]}$ and for each $i < j_k$, $\left(X^{(k)}\right)^{[i]} = \left(Y^{(k)}\right)^{[i]}$. So, $A(X) \leq A(Y)$;

4. If $X = Y$, then trivially $A(X) = A(Y)$.

Now we will prove that A_n is an injective function, for each $n \in \mathbb{N}^{+2}$ since $A_1 = Id_{\mathbb{H}^{(1)}}$. Let $n \in \mathbb{N}^{+2}$ and $X, Y \in \mathbb{H}^{(n)}$ such that $X \neq Y$. If $A_n(X) = A_n(Y)$ then, for each $i \in \mathbb{N}^+$, $A_n(X)^{[i]} = A_n(Y)^{[i]}$, i.e. $\left(X^{(k)}\right)^{[j]} = \left(Y^{(k)}\right)^{[j]}$. Observe that for each $l \in \mathbb{N}_n$ and $p \in \mathbb{N}$,

[3] Decimal expansions for real numbers is proper if there is no $m \in \mathbb{N}$ such that for each $n > m$, the n-th digit is always 9.

taking $i = pn + l$, we have that $k = l$ and $j = p + 1$. Therefore, for each $l \in \mathbb{N}_n$ and $p \in \mathbb{N}$, $\left(X^{(l)}\right)^{[p]} = \left(Y^{(l)}\right)^{[p]}$, i.e. $X^{(l)} = Y^{(l)}$. So, A_n is injective. The remaining of the proof follows straightforward from Theorem 4.3.2. □

Example 4.3.1 *Taking the function* $A\colon \mathbb{H} \to [0,1]$ *in Proposition 4.3.1 and the functions* $f, g\colon \mathbb{H} \to \mathbb{R}$ *given as* $f(X) = -\#X$ *and* $g(X) = \#X$*. Three incomparable pairs of* $\langle \mathbb{H}, \leq_R\rangle$*-order are presented in the following items:*

(i) Let $X = \{1/8, 2/3, 4/5\}$ *and* $Y = \{1/3, 3/4\}$ *be THFE. Then, we have that* $A(0.125, 0.\overline{6}, 0.8) = 0.1682605\overline{600} < 0.373530\overline{30} = A(0.\overline{3}, 0.75)$ *and therefore,* $X \prec_A^f Y$ *and* $X \prec_A^g Y$*.*

(ii) Let $X = \{0.\overline{2}, 0.\overline{5}, 0.8\overline{2}, 0.9\overline{5}\}, Y = \{0.28\overline{2}, 0.59\overline{5}\} \in \mathbb{H}$*. So, we obtain* $A(0.\overline{2}, 0.\overline{5}, 0.8\overline{2}, 0.9\overline{5}) = 0.258\overline{925} = A(0.28\overline{2}, 0.59\overline{5})$*. Therefore, we obtain that* $X \prec_A^f Y$ *since* $f(X) = -4 < -2 = f(Y)$ *and* $Y \prec_A^g X$*, because* $g(Y) = 2 < 4 = g(X)$*.*

(iii) Let $X = \{0.5, 0.7, 0.9\}$ *and* $Y = \{0.58, 0.69, 0.7\}$ *be THFE. Since* $A(X) = 0.579$ *and* $A(Y) = 0.56789$*, then* $Y \prec_A^f X$ *and* $Y \prec_A^g X$*.*

For each $n \in \mathbb{N}^+$, $i \in \mathbb{N}_n$ and $x \in [0,1]$, the following notation is introduced:

(i) $\tau_{n,i}(x) = 0.x^{[i]}x^{[n+i]}x^{[2n+i]}\ldots$; and

(ii) $\sigma_n(x) = \begin{cases} 1 & \text{if } x = 1 \\ arg\ \max_{k\in\mathbb{N}_n} \tau_{k,1}(x) < \tau_{k,2}(x) < \ldots < \tau_{k,k}(x) & \text{if } x < 1, \end{cases}$

(iii) $\mathbb{H}^{(n\downarrow)} = \bigcup_{k=1}^{n} \mathbb{H}^{(k)}$.

Lemma 4.3.1 *Let* $A : \mathbb{H} \to [0,1]$ *be a function defined in Proposition 4.3.1 and* $f^* : \mathbb{H} \to \mathbb{R}$ *be a function satisfying the condition stated in Theorem 4.3.2. For all* $n \in \mathbb{N}^+$*, the function* $A_{n\downarrow}^{(-1)} : [0,1] \to \mathbb{H}^{(n\downarrow)}$ *defined by*

$$A_{n\downarrow}^{(-1)}(x) = \left\{\tau_{\sigma_n(x),i}(x) : i \in \mathbb{N}_{\sigma_n(x)}\right\}, \tag{20}$$

verifies the following conditions:

1. *for each* $X \in \mathbb{H}$ *such that* $A(X) \neq 1$*,* $A_{\#X\downarrow}^{(-1)}(A(X)) = X$*;*

2. *for each* $x \in [0,1]$ *and* $n \in \mathbb{N}$*,* $\#A_{n\downarrow}^{(-1)}(x) = \sigma_n(x)$*;*

3. *for each* $x \in [0,1]$ *and* $n \in \mathbb{N}$*,* $A(A_{n\downarrow}^{(-1)}(x)) = x$*;*

4. *if* $x, y \in [0,1]$ *and* $x \leq y$*, then* $A_{n\downarrow}^{(-1)}(x) \preceq_A^{f^*} A_{n\downarrow}^{(-1)}(y)$*;*

5. *for each* $x \in [0,1]$ *and* $n \in \mathbb{N}$, $A^{(-1)}_{n\downarrow}(x) = \mathbf{0}_{\mathbb{H}} \Leftrightarrow x = 0$ *and* $A^{(-1)}_{n\downarrow}(x) = \mathbf{1}_{\mathbb{H}} \Leftrightarrow x = 1$;

6. *for each* $x \in [0,1]$ *, if* $m \leq n$ *then* $A^{(-1)}_{m\downarrow}(x) \preceq^{f^*}_{A} A^{(-1)}_{n\downarrow}(x)$.

Proof: Let $A : \mathbb{H} \to [0,1]$ and $f^* : \mathbb{H} \to \mathbb{R}$ be functions satisfying the same condition stated in Theorem 4.3.2. First note that $A^{(-1)}_{n\downarrow}$ is well defined.
(1.) Since, when $A(X) \neq 1$, $\sigma_{\#X}(A(X)) = \#X$, then it is immediate;
(2.) For each $x \in [0,1]$ and $n \in \mathbb{N}^{+}$, since by definition, $\tau_{\sigma_n(x),1}(x) < \tau_{\sigma_n(x),2}(x) < \ldots < \tau_{\sigma_n(x),\sigma_n(x)}(x)$, then $\#A^{(-1)}_{n\downarrow}(x) = \#\left\{\tau_{\sigma_n(x),i}(x) : i \in \mathbb{N}_{\sigma_n(x)}\right\} = \sigma_n(x)$;
(3.) Immediate;
(4.) Let $x, y \in [0,1]$ be such that $x < y$. From item 3, $A(A^{(-1)}_{n\downarrow}(x)) < A(A^{(-1)}_{n\downarrow}(y))$. Therefore, by definition of $\preceq^{f^*}_{A}$, $A^{(-1)}_{n\downarrow}(x) \prec^{f^*}_{A} A^{(-1)}_{n\downarrow}(y)$;
(5.) Immediate;
(6.) Straightforward from Eq. (20). So, one can conclude Lemma 4.3.1 is held. □

In order to reduce notation we consider $A^{(-1)}_{1\downarrow} \equiv A^{(-1)}_{1}$.

4.4 Chapter summary

In this chapter, we reported the notion of some typical hesitant fuzzy based on an admissible partial order $\leq$ related to the poset $(\mathbb{H}, \leq)$. Thus, we made use of total admissible $\langle \mathbb{H}, \preceq \rangle$-orders, a family of total orders that refine the partial $\langle \mathbb{H}, \leq_{RH} \rangle$-order. Then, here were introduced three new admissible orders in the typical hesitant fuzzy setting, which will allow us to present the notion of some typical hesitant connectives in the next chapters.

5 SOME CLASSES OF $\langle\mathbb{H},\preceq\rangle$-OPERATORS

In this section, the notion of some typical hesitant fuzzy connectives is presented, based on an admissible order $\preceq$ related to the poset $(\mathbb{H},\preceq)$. Main properties of $\langle\mathbb{H},\preceq\rangle$-negations and $\langle\mathbb{H},\preceq\rangle$-implication functions are analysed.

5.1 Typical hesitant fuzzy negations

In (SANTOS et al., 2014), it was presented the notion of typical hesitant fuzzy negations (THFN) using Xu-Xia-partial order and in (BEDREGAL et al., 2014), another order on $\mathbb{H}$ was considered for the THFN. As a more general setting, the concept of $\langle\mathbb{H},\preceq\rangle$-negation is presented, considering an arbitrary admissible order $\preceq$ on $\mathbb{H}$.

Definition 5.1.1 *Let $\preceq$ be admissible orders. A function $\mathcal{N}:\mathbb{H}\to\mathbb{H}$ is a THFN w.r.t $\langle\mathbb{H},\preceq\rangle$-order and called as $\langle\mathbb{H},\preceq\rangle$-negation $\mathcal{N}$ if, for all $X,Y\in\mathbb{H}$, the following properties are verified:*

$(\mathcal{N}1)$ $\mathcal{N}(\mathbf{0}_{\mathbb{H}})=\mathbf{1}_{\mathbb{H}}$ *and* $\mathcal{N}(\mathbf{1}_{\mathbb{H}})=\mathbf{0}_{\mathbb{H}}$ *(boundary conditions);*

$(\mathcal{N}2)$ *If $X\preceq Y$, then $\mathcal{N}(Y)\preceq\mathcal{N}(X)$ (antitonicity).*

Moreover, the $\langle\mathbb{H},\preceq\rangle$-negation $\mathcal{N}$ is strong if it is involutive:

$(\mathcal{N}3)$ $\mathcal{N}(\mathcal{N}(X))=X$, $\forall X\in\mathbb{H}$.

Additionally, an $\langle\mathbb{H},\preceq\rangle$-negation $\mathcal{N}$ is strictly decreasing meaning that

$(\mathcal{N}4)$ if $X\prec Y$, then $\mathcal{N}(X)\prec\mathcal{N}(Y)$.

Proposition 5.1.1 *If $\mathcal{N}:\mathbb{H}\to\mathbb{H}$ is a strong $\langle\mathbb{H},\preceq\rangle$-negation, then it is strictly decreasing, i.e. for each $X,Y\in\mathbb{H}$, if $X\prec Y$, then $\mathcal{N}(Y)\prec\mathcal{N}(X)$.*

Proof: Let $X,Y\in\mathbb{H}$ such that $X\prec Y$. By $(\mathcal{N}2)$, $\mathcal{N}(Y)\preceq\mathcal{N}(X)$. If $\mathcal{N}(Y)=\mathcal{N}(X)$, then $X=\mathcal{N}(\mathcal{N}(X))=\mathcal{N}(\mathcal{N}(Y))=Y$, which is a contradiction. So, $\mathcal{N}(Y)\prec\mathcal{N}(X)$. □

Corollary 5.1.1 *If $\mathcal{N}:\mathbb{H}\to\mathbb{H}$ is a strong $\langle\mathbb{H},\preceq\rangle$-negation and $\alpha\in\mathbb{H}$, such that $\mathcal{N}(\alpha)=\alpha$. Then, for all $X<\alpha<Y$, we have that $\mathcal{N}(Y)\prec\alpha\prec\mathcal{N}(X)$.*

Proof: Straightforward from Proposition 5.1.1. □

Proposition 5.1.2 *For an $\langle \mathbb{H}, \preceq \rangle$-negation $\mathcal{N}$, $\mathcal{N}_\perp \preceq \mathcal{N} \preceq \mathcal{N}_\top$, whenever*

$$\mathcal{N}_\perp(X) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \textit{if } X = \mathbf{0}_{\mathbb{H}}, \\ \mathbf{0}_{\mathbb{H}}, & \textit{otherwise;} \end{cases} \quad \textit{and} \quad \mathcal{N}_\top(X) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \textit{if } X \prec \mathbf{1}_{\mathbb{H}}, \\ \mathbf{0}_{\mathbb{H}}, & \textit{otherwise.} \end{cases}$$

Proof: Straightforward from Definition 5.1.1. □

Proposition 5.1.3 *Each $\langle \mathbb{H}, \preceq \rangle$-negation $\mathcal{N}$ is an $\langle \mathbb{H}, \preceq \rangle$-join morphism and an $\langle \mathbb{H}, \preceq \rangle$-meet morphism, i.e. for each $X, Y \in \mathbb{H}$, the following expressions hold $\mathcal{N}(X \vee Y) = \mathcal{N}(X) \wedge \mathcal{N}(Y)$ and $\mathcal{N}(X \wedge Y) = \mathcal{N}(X) \vee \mathcal{N}(Y)$, where $\vee$ and $\wedge$ are the maximum and minimum w.r.t. $\preceq$-order.*

Proof: Since $\preceq$ is a linear order, then either $X \preceq Y$ or $Y \preceq X$. Without loss of generality, assume that $X \preceq Y$ and, hence, $\mathcal{N}(Y) \preceq \mathcal{N}(X)$, $X \vee Y = Y$ and $X \wedge Y = X$. So, $\mathcal{N}(X \vee Y) = \mathcal{N}(Y) = \mathcal{N}(X) \wedge \mathcal{N}(Y)$ and $\mathcal{N}(X \wedge Y) = \mathcal{N}(X) = \mathcal{N}(X) \vee \mathcal{N}(Y)$. □

5.2 Generating $\langle \mathbb{H}, \preceq \rangle$-negations from fuzzy negations

Firstly, we present a simple method to generate $\langle \mathbb{H}, \preceq \rangle$-negations w.r.t. both admissible orders, $\preceq_{Lex1}$- and $\preceq_{Lex2}$-order, which are given by Eq.(17) and Eq.(18), respectively.

Proposition 5.2.1 *Let $N : [0,1] \to [0,1]$ be a fuzzy negation. Then, the functions $\widetilde{N}, \widetilde{\widetilde{N}} : \mathbb{H} \to \mathbb{H}$ respectively defined as $\widetilde{N}(X) = \left\{N\left(X^{(1)}\right)\right\}$ and $\widetilde{\widetilde{N}}(X) = \left\{N\left(X^{(\#X)}\right)\right\}, \forall X \in \mathbb{H}$, are $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-negations.*

Proof: The proof that function $\widetilde{N}$ is an $\langle \mathbb{H}, \preceq_{Lex1} \rangle-$negation is presented. The other one can be analogously constructed. Let $\preceq_{Lex1}$ be an admissible order given by Eq.(17). So, we have:
$(\mathcal{N}1)$ $\widetilde{N}(\mathbf{0}_{\mathbb{H}}) = \{N(0)\} = \mathbf{1}_{\mathbb{H}}$ and $\widetilde{N}(\mathbf{1}_{\mathbb{H}}) = \{N(1)\} = \mathbf{0}_{\mathbb{H}}$;
$(\mathcal{N}2)$ If $X \preceq_{Lex1} Y$, then $X^{(1)} \leq Y^{(1)}$ and $\widetilde{N}(Y) = \{N(Y^{(1)})\} \preceq_{Lex1} \{N(X^{(1)})\} = \widetilde{N}(X)$.

Since the case of $\preceq_{Lex2}$ is analogous, therefore, Proposition 5.2.1 is verified. □

Corollary 5.2.1 *Let $N : [0,1] \to [0,1]$ be a negation. Then, for all $X \in \mathbb{H}$, $\widetilde{\widetilde{N}}(X) \leq_{RH} \widetilde{N}(X)$.*

Example 5.2.1 *Considering the results from Proposition 5.2.1, and $\widetilde{N}(\widetilde{\widetilde{N}})$ be the standard $\langle [0,1], \leq \rangle$-negation defined here as $1 - x, \forall x \in [0,1]$, the following holds:*
(i) For $X' = \{0.\overline{3}, 0.445, 1.0\}$, $\widetilde{N}(X') = \{0.\overline{6}\}$ and $\widetilde{\widetilde{N}}(X') = \mathbf{0}_{\mathbb{H}}$;
(ii) For $X'' = \{0.1, 0.35, 0.9\}$, $\widetilde{N}(X'') = \{0.9\}$ and $\widetilde{\widetilde{N}}(X'') = \{0.1\}$.

Remark 5.2.1 *Despite of N being a strong fuzzy negation, the $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$- negations generated from N do not verify the involutive property. By taking $X \in \mathbb{H}$ such that $\#X \in \mathbb{N}_2$ and a strong negation N, then $\widetilde{N}(\widetilde{N}(X)) = \{N(N(X^{(1)}))\} = \{X^{(1)}\} \subset X$. So, $\widetilde{N}$ is a non-strong $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-negation. The same analysis can be extended to the $\widetilde{\widetilde{N}}$ function.*

Now, a methodology to obtain strong fuzzy negations w.r.t. $\langle \mathbb{H}, \preceq_A^f \rangle$-order is presented.

Theorem 5.2.1 *Let $N : [0,1] \to [0,1]$ be a strictly decreasing fuzzy negation, $A : \mathbb{H} \to [0,1]$ as required in Theorem 4.3.2 and $f^* : \mathbb{H} \to \mathbb{R}$ be the function $f^*(X) = -\#X$ for each $X \in \mathbb{H}$. The function $\mathcal{N}_{A,f^*} : \mathbb{H} \to \mathbb{H}$, given by*

$$\mathcal{N}_{A,f^*}(X) = A^{(-1)}_{\#X\downarrow}(N(A(X))), \tag{21}$$

where $A_n^{(-1)}$ is the function defined in Lemma 4.3.1, is an $\langle \mathbb{H}, \preceq_A^f \rangle$-negation. In addition,

1. *$\mathcal{N}_{A,f^*}$ has an inverse function;*
2. *If N is a strong fuzzy negation, then $\mathcal{N}_{A,f^*}$ is also strong.*

Proof: Trivially, $\mathcal{N}_{A,f^*}(\mathbf{0}_\mathbb{H}) = \mathbf{1}_\mathbb{H}$ and $\mathcal{N}_{A,f^*}(\mathbf{1}_\mathbb{H}) = \mathbf{0}_\mathbb{H}$. If $X \preceq_A^{f^*} Y$. So, it holds that:
(i) if $A(X) < A(Y)$, then the next results hold:

$$\begin{aligned} A(X) < A(Y) &\Leftrightarrow N(A(Y)) < N(A(X))), \text{ by } N \text{ being strictly decreasing;} \\ &\Leftrightarrow A(A^{(-1)}_{\#Y\downarrow}(N(A(Y)))) < A(A^{(-1)}_{\#X\downarrow}(N(A(X)))), \text{ by Lemma 4.3.1;} \\ &\Leftrightarrow A(\mathcal{N}_{A,f^*}(Y)) < A(\mathcal{N}_{A,f^*}(X)), \text{ by Eq. (21).} \end{aligned}$$

So, $\mathcal{N}_{A,f^*}(Y) \prec_A^{f^*} \mathcal{N}_{A,f^*}(X)$.
(ii) if $A(X) = A(Y)$ and $f^*(X) \leq f^*(Y)$, then it holds that:
(a) $N(A(X)) = N(A(Y))$ and, by Lemma 4.3.1, $A(\mathcal{N}_{A,f^*}(X)) = A(A^{(-1)}_{\#X\downarrow}(N(A(X)))) = N(A(X)) = N(A(Y)) = A(A^{(-1)}_{\#Y\downarrow}(N(A(Y)))) = A(\mathcal{N}_{A,f^*}(Y))$;
(b) Since $f^*(X) \leq f^*(Y)$ then $\#X > \#Y$. So, because $N(A(X)) = N(A(Y))$ and by Lemma 4.3.1, $A^{(-1)}_{\#Y\downarrow}(N(A(Y))) \preceq_A^{f^*} A^{(-1)}_{\#X\downarrow}(N(A(X)))$. Therefore, $\mathcal{N}_{A,f^*}(Y) \preceq_A^{f^*} \mathcal{N}_{A,f^*}(X)$.

Hence, $\mathcal{N}_{A,f^*}(Y) \preceq_A^{f^*} \mathcal{N}_{A,f^*}(X)$, and the function $\mathcal{N}_{A,f^*}$ is an $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-negation.
In order to prove item 1, since N is strictly decreasing then it is invertible, with N^{-1} as its inverse. Let $\mathcal{N}^{(-1)}_{A,\hat{f}^*} : \mathbb{H} \to \mathbb{H}$, defined by $\mathcal{N}^{(-1)}_{A,f^*}(X) = A^{(-1)}_{\#X\downarrow}(N^{-1}(A(X)))$, for each $X \in \mathbb{H}$. Since, N^{-1} is also a strictly decreasing fuzzy negation, then, by the previous proof, $\mathcal{N}^{(-1)}_{A,f^*}$ is also an $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-negation. Moreover, from Lemma 4.3.1, we have that $\mathcal{N}^{(-1)}_{A,f^*}(\mathcal{N}_{A,f^*}(X)) =$

$A^{(-1)}_{\#X\downarrow}(N^{-1}(A(A^{(-1)}_{\#X\downarrow}(N(A(X)))))) = X$, and analogously, $\mathcal{N}_{A,f^*}(\mathcal{N}^{(-1)}_{A,f^*}(X)) = A^{(-1)}_{\#X\downarrow}(N(A(A^{(-1)}_{\#X\downarrow}(N^{-1}(A(X)))))) = X$. Consequently, $\mathcal{N}_{A,f^*}$ is invertible. The proof of item 2 can be obtained analogously. Therefore, Theorem 5.2.1 is verified. □

Example 5.2.2 *Let N_S be the standard $\langle[0,1],\leq\rangle$-negation defined here as*

$$N_S(x) = 0.\overline{9} - x, \forall x \in [0,1].$$

Consider $f^* : \mathbb{H} \to \mathbb{R}$, $f^*(X) = -\#X$ and A as the entangle order given in Prop. 4.3.1. By Theorem 5.2.1, the function $\mathcal{N}_{SA,f^*}$, given in Eq. (21) is a strong $\langle\mathbb{H}, \preceq^{f^*}_A\rangle$-negation.

See, e.g., the following results:

(1) For $X'' = \{0.1, 0.35, 0.9\}$, following the above methodology: $\mathcal{N}_{SA,f^*}(X'') = (A^{(-1)}_{3\downarrow} \circ N_S \circ A)(\{0.1, 0.35, 0.9\}) = (A^{(-1)}_{3\downarrow} \circ N_S)(0.139050) = A^{(-1)}_{3\downarrow}(0.86094\overline{9})$. Now, we need to consider the following calculus.
For $k = 3$ we obtain that:
(i) $\tau(3,1) = 0.8\overline{9} \equiv 0.9$;
(ii) $\tau(3,2) = 0,64\overline{9} \equiv 0,65$; and
(iii) $\tau(3,3) = 0.0\overline{9} \equiv 0.1$.
Therefore, $\tau(3,1) > \tau(3,2) > \tau(3,3)$ and $\sigma_3(0.86094\overline{9}) \neq 3$. Analogously, for $k = 2$. Meaning that $\sigma_3(0.86094\overline{9}) = 1$ implying $\mathcal{N}_{SA,f^*}(X'') = \{0.86095\}$.

5.3 Typical hesitant aggregation functions

In (BEDREGAL et al., 2014a), an extended notion of n-ary aggregation functions for THFE using one of the partial orders is proposed by the authors. In the following, the definition of $\langle\mathbb{H},\preceq\rangle$-aggregation function generalizes that notion by considering an admissible $\langle\mathbb{H},\preceq\rangle$-order.

Definition 5.3.1 *A function $\mathcal{A}\colon \mathbb{H}^n \to \mathbb{H}$ is an n-ary typical hesitant aggregation function w.r.t. the admissible total $\langle\mathbb{H},\preceq\rangle$-order, called as $\langle\mathbb{H},\preceq\rangle$-aggregation, if the following properties are verified:*

($\mathcal{A}1$) *$\mathcal{A}(X_1,\ldots,X_n) \preceq \mathcal{A}(Y_1,\ldots,Y_n)$, when $X_i \preceq Y_i$, for all $i \in \mathbb{N}_n$ (isotonicity);*

($\mathcal{A}2$) *$\mathcal{A}(\mathbf{0}_\mathbb{H},\ldots,\mathbf{0}_\mathbb{H}) = \mathbf{0}_\mathbb{H}$ and $\mathcal{A}(\mathbf{1}_\mathbb{H},\ldots,\mathbf{1}_\mathbb{H}) = \mathbf{1}_\mathbb{H}$ (boundary conditions).*

In the following, some extra properties can be satisfied for some (but not all) typical hesitant extended aggregation functions. For example, $\mathcal{A}$ satisfies:

($\mathcal{A}3$) symmetry if for each permutation ( ) : $\mathbb{N}_n \to \mathbb{N}_n$,

$$\mathcal{A}(X_1,\ldots,X_n) = \mathcal{A}(X_{(1)},\ldots,X_{(n)}).$$

An $\langle \mathbb{H}, \preceq \rangle$-aggregation function $\mathcal{A}$ is a conjunctive (disjunctive) function if, for each $i \in \mathbb{N}_n$, $X_i \preceq \mathcal{A}(X_1, \ldots, X_n)$ ($X_i \succeq \mathcal{A}(X_1, \ldots, X_n)$). Moreover, when $\mathcal{A}$ is disjunctive (conjunctive) and bivariate, we respectively have that:

$(\mathcal{A}4)$ $\mathcal{A}(\mathbf{1}_{\mathbb{H}}, X) = \mathcal{A}(X, \mathbf{1}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$, for all $X \in \mathbb{H}$;

$(\mathcal{A}5)$ $\mathcal{A}(\mathbf{0}_{\mathbb{H}}, X) = \mathcal{A}(X, \mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$, for all $X \in \mathbb{H}$.

In particular, from $(\mathcal{A}4)$ and $(\mathcal{A}5)$ we have the corresponding properties:

$(\mathcal{A}4a)$ $\mathcal{A}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = \mathcal{A}(\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$;

$(\mathcal{A}5a)$ $\mathcal{A}(\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}) = \mathcal{A}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$.

Definition 5.3.2 *A function $\mathcal{A} : \bigcup_{n=1}^{n} \mathbb{H}^n \to \mathbb{H}$ is a typical hesitant extended aggregation function if, for each $n \in \mathbb{N}^{+2}$, $\mathcal{A} \upharpoonright \mathbb{H}^n$ is an $\langle \mathbb{H}, \preceq \rangle$-aggregation function and $\mathcal{A}(X) = X$, for each $X \in \mathbb{H}$.*

Any $\langle \mathbb{H}, \preceq \rangle$-order related to a typical hesitant extended aggregation function (THEA) is identified with a family of functions $(\mathcal{A}_n)_{n \in \mathbb{N}^{+2}}$ such that $\mathcal{A}_n$ is an $\langle \mathbb{H}, \preceq \rangle$- aggregation.

Remark 5.3.1 *Let $\mathcal{A}$ be a binary $\langle \mathbb{H}, \preceq \rangle$-aggregation function, a THEA $\mathcal{A}'$ is given as follows:*

$$\mathcal{A}'(X) = X; \text{ and } \mathcal{A}'(X_1, \ldots, X_n) = \mathcal{A}(X_1, \mathcal{A}'(X_2, \ldots, X_n)), \forall n \in \mathbb{N}^{+2}. \qquad (22)$$

For THEA, the following properties can also be considered:

$(\mathcal{A}6)$ invariance of replication if:

$$\mathcal{A}(X_1, \ldots, X_n) = \mathcal{A}(X_1, \ldots, X_n, \ldots, X_1, \ldots, X_n);$$

$(\mathcal{A}7)$ invariance for $\mathbf{1}_{\mathbb{H}}$ if, for each $i = 1, \ldots, n$:

$$\mathcal{A}(X_1, \ldots, X_n) = \mathcal{A}(X_1, \ldots, X_i, \mathbf{1}_{\mathbb{H}}, X_{i+1}, \ldots, X_n);$$

$(\mathcal{A}8)$ idempotence for all $X \in \mathbb{H}$:

$$\mathcal{A}(X, \ldots, X) = X.$$

5.4 Typical hesitant triangular norms

The extension of the notion of t-norms for typical hesitant fuzzy elements was presented in (BEDREGAL et al., 2014a), taking into account the partial order proposed in that paper. The following definition generalizes this notion by considering admissible orders on $\mathbb{H}$.

Definition 5.4.1 *Let* $\mathcal{T}\colon \mathbb{H}^2 \to \mathbb{H}$ *and let the admissible* $\langle \mathbb{H}, \preceq \rangle$*-order on* $\mathbb{H}$*.* $\mathcal{T}$ *is a typical hesitant triangular norm w.r.t.* $\langle \mathbb{H}, \preceq \rangle$*-order, or* $\langle \mathbb{H}, \preceq \rangle$*-t-norm in short, if*

$(\mathcal{T}1)$ *It is commutative:* $\mathcal{T}(X, Y) = \mathcal{T}(Y, X)$*;*

$(\mathcal{T}2)$ *It is associative:* $\mathcal{T}(X, \mathcal{T}(Y, Z)) = \mathcal{T}(\mathcal{T}(X, Y), Z)$*;*

$(\mathcal{T}3)$ *It is monotonic, i.e., if* $X \preceq Y$ *then* $\mathcal{T}(X, Z) \preceq \mathcal{T}(Y, Z)$*; and*

$(\mathcal{T}4)$ $\mathbf{1}_{\mathbb{H}}$ *is the neutral element:* $\mathcal{T}(X, \mathbf{1}_{\mathbb{H}}) = X$*.*

Remark 5.4.1 *For all* $\langle \mathbb{H}, \preceq \rangle$*-t-norm,* $\mathcal{T}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = \mathcal{T}(\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$*.*

Example 5.4.1 *By taking* $n = \max\{\#X, \#Y\}$*, the corresponding extension* T_P, T_M *and* T_{LK} *given as follows:*

i. $\mathcal{T}_P(X, Y) = A_{n\downarrow}^{(-1)}(T_P(A(X), A(Y)))$*;*

ii. $\mathcal{T}_M(X, Y) = A_{n\downarrow}^{(-1)}(T_M(A(X), A(Y)))$*;*

iii. $\mathcal{T}_{LK}(X, Y) = A_{n\downarrow}^{(-1)}(T_{LK}(A(X), A(Y)))$*,*

are typical hesitant t-norms w.r.t. the admissible order $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$*. Moreover, we also have that:*

i. $\mathcal{T}_P(X, Y) = \{x \cdot y \mid x \in X, y \in Y\}$,

ii. $\mathcal{T}_M(X, Y) = \{\min\{x, y\} \mid x \in X, y \in Y\}$,

iii. $\mathcal{T}_L(X, Y) = \{\max\{x + y - 1, 0\} \mid x \in X, y \in Y\}$*,*

are typical hesitant t-norms w.r.t. the admissible $\langle \mathbb{H}, \preceq_{Lex1} \rangle$*- and* $\langle \mathbb{H}, \preceq_{Lex2} \rangle$*-orders.*

5.5 Generating $\langle \mathbb{H}, \preceq \rangle$-aggregations from aggregations

The idea of $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-, $\langle \mathbb{H}, \preceq_{Lex2} \rangle$ and $\langle \mathbb{H}, \preceq_A^f \rangle$-OWA, are presented in this section as the ordered weighted average aggregations, considering the admissible $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-, $\langle \mathbb{H}, \preceq_{Lex2} \rangle$- and $\langle \mathbb{H}, \preceq_A^f \rangle$- orders, respectively.

5.5.1 Generating $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-aggregations from aggregations

The concept of $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-OWA is presented as the ordered weighted average aggregations, considering the admissible $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-orders.

Proposition 5.5.1 *Let $M : [0,1]^n \to [0,1]$ be a strict aggregation and $m = \max\{\#X_i : i \in \mathbb{N}_n\}$. The function $\mathcal{M}^{Lex1} : \mathbb{H}^n \to \mathbb{H}$, defined as*

$$\mathcal{M}^{Lex1}(X_1, \ldots, X_n) = \left\{ M\left(X_1^{(1,m)}, \ldots, X_n^{(1,m)}\right), \ldots, M\left(X_1^{(m,m)}, \ldots, X_n^{(m,m)}\right) \right\}, \quad (23)$$

is a THEA where $X_i^{(j,m)} = \begin{cases} X_i^{(j)}, & \textit{if } j \leq \#X_i \\ X_i^{(\#X_i)}, & \textit{otherwise}. \end{cases}$

Proof: Trivially, $\mathcal{M}^{Lex1}(\mathbf{0}_{\mathbb{H}}, \ldots, \mathbf{0}_{\mathbb{H}}) = \left\{ M\left(\mathbf{0}_{\mathbb{H}}^{(1,1)}, \ldots, \mathbf{0}_{\mathbb{H}}^{(1,1)}\right) \right\} = \{M(0, \ldots, 0)\} = \{0\} = \mathbf{0}_{\mathbb{H}}$ and $\mathcal{M}^{Lex1}(\mathbf{1}_{\mathbb{H}}, \ldots, \mathbf{1}_{\mathbb{H}}) = \left\{ M\left(\mathbf{1}_{\mathbb{H}}^{(1,1)}, \ldots, \mathbf{1}_{\mathbb{H}}^{(1,1)}\right) \right\} = \{M(1, \ldots, 1)\} = \{1\} = \mathbf{1}_{\mathbb{H}}$.
Let $X_1, \ldots, X_n, Y \in \mathbb{H}$ such that $X_i \prec_{Lex1} Y$ for some $i \in \mathbb{N}_n$. Then, either (1) there exist $k \leq \min\{\#X_i, \#Y\}$ such that $X_i^{(k)} < Y^{(k)}$ and $X_i^{(j)} = Y^{(j)}$ for each $j < k$, or (2) $\#X_i \leq \#Y$ and $X_i^{(j)} = Y^{(j)}$ for each $j \leq \#X_i$.
Case (1): Clearly, $M\left(X_1^{(j,m)}, \ldots, X_n^{(j,m)}\right) = M\left(X_1^{(j,m)}, \ldots, X_{i-1}^{(j,m)}, Y^{(j,m)}, X_{i+1}^{(j,m)}, \ldots, X_n^{(j,m)}\right)$ for each $j < k$ and, because M is strictly increasing:
(i) $M\left(X_1^{(k,m)}, \ldots, X_n^{(k,m)}\right) < M\left(X_1^{(k,m)}, \ldots, X_{i-1}^{(k,m)}, Y^{(k,m)}, X_{i+1}^{(k,m)}, \ldots, X_n^{(k,m)}\right)$, and
(ii) $\mathcal{M}^{Lex1}(X_1, \ldots, X_n)^{(j,m)} = M\left(X_1^{(j,m)}, \ldots, X_n^{(j,m)}\right)$ for each $j \in \mathbb{N}_m$.
Therefore, $\mathcal{M}^{Lex1}(X_1, \ldots, X_n) \preceq_{Lex1} \mathcal{M}^{Lex1}(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n)$.
Case (2): Likewise, $\forall j \in \mathbb{N}_{\#X_i}$ we have that

$$\begin{aligned}
\mathcal{M}^{Lex1}(X_1, \ldots, X_n)^{(j,m)} &= M\left(X_1^{(j,m)}, \ldots, X_n^{(j,m)}\right) \\
&= M\left(X_1^{(j,m)}, \ldots, X_{i-1}^{(j,m)}, Y^{(j,m)}, X_{i+1}^{(j,m)}, \ldots, X_n^{(j,m)}\right) \\
&= \mathcal{M}^{Lex1}(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n)^{(j,m)} \\
&= M\left(X_1^{(j,m)}, \ldots, X_{i-1}^{(j,m)}, Y^{(j,m)}, X_{i+1}^{(j,m}, \ldots, X_n^{(j,m)}\right) \\
&\leq \mathcal{M}(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n)^{(j,m)}.
\end{aligned}$$

So, $\mathcal{M}^{Lex1}(X_1, \ldots, X_n) \preceq_{Lex1} \mathcal{M}^{Lex1}(X_1, \ldots, X_{i-1}, Y, X_{i+1}, \ldots, X_n)$. And, Prop. 5.5.1 holds. □

In the next corollary we describe the expression of an $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-OWA operator.

Corollary 5.5.1 *Let $OWA_\omega : [0,1]^n \to [0,1]$ be the EAF defined by $OWA_\omega(x_1, \ldots, x_n) = \sum_{i=1}^{n} w_i x_{\sigma(i)}$, w.r.t. a positive weighing vector $\omega = (w_1, w_2, \ldots w_n) \in [0,1]^n$. For $m =$*

$\max\{\#X_i : i \in \mathbb{N}_n\}$*, the function* $\mathcal{OWA}_{\omega}^{Lex1} : \mathbb{H}^n \to \mathbb{H}$*, given as*

$$\mathcal{OWA}_{\omega}^{Lex1}(X_1, \ldots, X_n) = \left\{OWA_{\omega}\left(X_1^{(1,m)}, \ldots, X_n^{(1,m)}\right), \ldots, OWA_{\omega}\left(X_1^{(m,m)}, \ldots, X_n^{(m,m)}\right)\right\}, \tag{24}$$

is an $\langle \mathbb{H}, \preceq_{Lex1}\rangle$*-aggregation, whenever* $X_i^{(j,m)} = \begin{cases} X_i^{(j)}, & \text{if } j \leq \#X_i \\ X_i^{(\#X_i)}, & \text{otherwise.} \end{cases}$

Proof: Straight from Prop. 5.5.1, as an OWA is strictly increasing and ω is positive (BELIAKOV; PRADERA; CALVO, 2007). □

Example 5.5.1 *Considering the following tuples of THFS in* $\mathbb{H}^4$ *and the weighing vector* $\omega = (0.1, 0.2, 0.3, 0.4)$*:*

$X_1 = (\{0.4\}, \{0.7\}, \mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}})$; $X_2 = (\{0.75\}, \{0.85\}, \mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}})$;
$X_3 = (\{0.45\}, \{0.7\}, \mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}})$; $X_4 = (\{0.7\}, \{0.8\}, \mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}})$;
$X_5 = (\{0.39, 0.44, 0.45\}, \{0.61, 0.75\}, \mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}})$; $X_6 = (\mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}, \{0.6, 0.8, 0.9\}, \{0.4, 0.7, 0.8\})$.

Then, we obtain that the following results:

1. $\mathcal{OWA}_{\omega}^{Lex1}(X_1) = \{0.1\cdot0.4+0.2\cdot0.7+0.3\cdot1+0.4\cdot1\} = \{0.04+0.14+0.3+0.4\} = \{0.88\}$;
2. $\mathcal{OWA}_{\omega}^{Lex1}(X_2) = \{0.1\cdot0.75+0.2\cdot0.85+0.3\cdot1+0.4\cdot1\} = \{0.075+0.17+0.3+0.4\} = \{0.945\}$;
3. $\mathcal{OWA}_{\omega}^{Lex1}(X_3) = \{0.1\cdot0.45+0.2\cdot0.7+0.3\cdot1+0.4\cdot1\} = \{0.045+0.14+0.3+0.4\} = \{0.885\}$;
4. $\mathcal{OWA}_{\omega}^{Lex1}(X_4) = \{0.1\cdot0.7+0.2\cdot0.8+0.3\cdot1+0.4\cdot1\} = \{0.07+0.16+0.3+0.4\} = \{0.93\}$;
5. $\mathcal{OWA}_{\omega}^{Lex1}(X_5) = \{0.1 \cdot 0.39 + 0.2 \cdot 0.61 + 0.3 \cdot 1.0 + 0.4 \cdot 1.0, 0.1 \cdot 0.44 + 0.2 \cdot 0.75 + 0.3 \cdot 1.0 + 0.4 \cdot 1.0, 0.1 \cdot 0.45 + 0.2 \cdot 0.75 + 0.3 \cdot 1.0 + 0.4 \cdot 1.0\} = \{0.861, 0.894, 0.895\}$;
6. $\mathcal{OWA}_{\omega}^{Lex1}(X_6) = \{0.1 \cdot 0.4 + 0.2 \cdot 0.6 + 0.3 \cdot 1.0 + 0.4 \cdot 1.0, 0.1 \cdot 0.7 + 0.2 \cdot 0.8 + 0.3 \cdot 1.0 + 0.4 \cdot 1.0, 0.1 \cdot 0.8 + 0.2 \cdot 0.9 + 0.3 \cdot 1.0 + 0.4 \cdot 1.0\} = \{0.86, 0.93, 0.96\}$.

5.5.2 Generating $\langle \mathbb{H}, \preceq_{Lex2}\rangle$-aggregations from aggregations

The concept of $\langle \mathbb{H}, \preceq_{Lex2}\rangle$-OWA is presented as the ordered weighted average aggregations, considering the admissible $\langle \mathbb{H}, \preceq_{Lex2}\rangle$-orders.

Proposition 5.5.2 *Let* $M : [0,1]^n \to [0,1]$ *be a strict aggregation function. When* $m = \max\{\#X_i : i \in \mathbb{N}_n\}$*, the function* $\mathcal{M}^{Lex2} : \mathbb{H}^n \to \mathbb{H}$*, defined as follows:*

$$\mathcal{M}^{Lex2}(X_1, \ldots, X_n) = \left\{M\left(X_1^{(1,m)}, \ldots, X_n^{(1,m)}\right), \ldots, M\left(X_1^{(m,m)}, \ldots, X_n^{(m,m)}\right)\right\}, \tag{25}$$

is an $\langle \mathbb{H}, \preceq_{Lex2}\rangle$*-aggregation operator, where* $X_i^{(j,m)} = \begin{cases} X_i^{\#X_i - m + j}, & \text{if } j > m - \#X_i \\ X_i^{(1)}, & \text{otherwise.} \end{cases}$

Proof: Analogous to the proof of Proposition 5.5.1. □

In the next corollary, we describe the expression of an $\langle\mathbb{H}, \preceq_{Lex2}\rangle$-OWA operator.

Corollary 5.5.2 *Let $OWA_\omega : [0,1]^n \to [0,1]$ be the ordered weighted average operator (OWA) w.r.t. the positive weighing vector $\omega = (w_1, w_2, \ldots w_n)$. When $m = \max\{\#X_i : i \in \mathbb{N}_n\}$, the function $\mathcal{OWA}^{Lex2}_\omega : \mathbb{H}^n \to \mathbb{H}$, defined by*

$$\mathcal{OWA}^{Lex2}_\omega(X_1, \ldots, X_n) = \left\{OWA_\omega\left(X_1^{(1,m)}, \ldots, X_n^{(1,m)}\right), \ldots, OWA_\omega\left(X_1^{(m,m)}, \ldots, X_n^{(m,m)}\right)\right\}, \tag{26}$$

is an $\langle\mathbb{H}, \preceq_{Lex2}\rangle$-aggregation operator, where $X_i^{(j,m)} = \begin{cases} X_i^{\#X_i - m + j}, & \text{if } j > m - \#X_i \\ X_i^{(1)}, & \text{otherwise.} \end{cases}$

Proof: Analogously, considering the OWA as a strictly increasing function whenever ω is positive (see (BELIAKOV; PRADERA; CALVO, 2007, p.69)) then, from Proposition 5.5.1, Corollary 5.5.2 holds. □

Example 5.5.2 *Considering the tuples of THFS in $\mathbb{H}^4$ and the weighting vector ω presented in Example 5.5.1. Then, we obtain that the following results:*

1. $\mathcal{OWA}^{Lex2}_\omega(X_1) = \{0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.7 + 0.4\cdot 0.4\} = \{0.1 + 0.2 + 0.21 + 0.16\} = \{0.67\}$;
2. $\mathcal{OWA}^{Lex2}_\omega(X_2) = \{0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.85 + 0.4\cdot 0.75\} = \{0.1 + 0.2 + 0.255 + 0.3\} = \{0.855\}$;
3. $\mathcal{OWA}^{Lex2}_\omega(X_3) = \{0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.7 + 0.4\cdot 0.45\} = \{0.1 + 0.2 + 0.21 + 0.18\} = \{0.69\}$;
4. $\mathcal{OWA}^{Lex2}_\omega(X_4) = \{0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.8 + 0.4\cdot 0.7\} = \{0.1 + 0.2 + 0.24 + 0.28\} = \{0.82\}$;
5. $\mathcal{OWA}^{Lex2}_\omega(X_5) = \{0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.75 + 0.4\cdot 0.45, 0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.61 + 0.4\cdot 0.44, 0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.61 + 0.4\cdot 0.39\} = \{0.705, 0.659, 0.639\}$;
6. $\mathcal{OWA}^{Lex2}_\omega(X_6) = \{0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.9 + 0.4\cdot 0.8, 0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.8 + 0.4\cdot 0.7, 0.1\cdot 1 + 0.2\cdot 1 + 0.3\cdot 0.6 + 0.4\cdot 0.4\} = \{0.89, 0.82, 0.64\}$.

5.5.3 Generating $\langle\mathbb{H}, \preceq^f_A\rangle$-aggregations from fuzzy aggregation functions

In this section, we present the concept of $\langle\mathbb{H}, \preceq^f_A\rangle$-OWA, as the ordered weighted average aggregation, considering the admissible $\langle\mathbb{H}, \preceq^f_A\rangle$-order.

Theorem 5.5.1 *Let $M\colon [0,1]^n \to [0,1]$ be an strictly increasing aggregation function, A and f^* be functions verifying the conditions of Theorem 4.3.2. The function $\mathcal{M}_{A,f^*} : \mathbb{H}^n \to \mathbb{H}$, given as*

$$\mathcal{M}_{A,f^*}(X_1, \ldots, X_n) = A^{(-1)}_{m\downarrow}\left(M(A(X_1), \ldots, A(X_n))\right), \tag{27}$$

is an $\langle\mathbb{H}, \preceq^{f^*}_A\rangle$-aggregation function, where $m = \max\{\#X_i : i \in \mathbb{N}_n\}$. In addition, if M verifies A_i then $\mathcal{M}_{A,f^*}$ satisfies $\mathcal{A}_i$, for $i \in \mathbb{N}_6$.

Proof: Trivially, $\mathcal{M}_{A,f^*}(\mathbf{0}_{\mathbb{H}},\ldots,\mathbf{0}_{\mathbb{H}}) = A_1^{(-1)}(M(A(\mathbf{0}_{\mathbb{H}}),\ldots,A(\mathbf{0}_{\mathbb{H}}))) = A_1^{(-1)}(M(0,\ldots,0)) = \mathbf{0}_{\mathbb{H}}$ and, besides, $\mathcal{M}_{A,f^*}(\mathbf{1}_{\mathbb{H}},\ldots,\mathbf{1}_{\mathbb{H}}) = A_1^{(-1)}(M(A(\mathbf{1}_{\mathbb{H}}),\ldots,A(\mathbf{1}_{\mathbb{H}}))) = A_1^{(-1)}(M(1,\ldots,1)) = \mathbf{1}_{\mathbb{H}}$. Let $X_1,\ldots,X_n,Y \in \mathbb{H}$, such that $X_i \prec_A^{f^*} Y$, for some $i \in \mathbb{N}_n$. Then, either $A(X_i) < A(Y)$ or $A(X_i) = A(Y)$ and $f^*(X_i) < f^*(Y)$. In case $A(X_i) < A(Y)$, then because M is strictly increasing, $M(A(X_1),\ldots,A(X_n)) < M(A(X_1),\ldots,A(X_{i-1}),A(Y),A(X_{i+1}),\ldots,A(X_n))$ and therefore, by Lemma 4.3.1, $\mathcal{M}_{A,f^*}(X_1,\ldots,X_n) \prec_A^{f^*} \mathcal{M}_{A,f^*}(X_1,\ldots,X_{i-1},Y,X_{i+1},\ldots,X_n)$. For the case that $A(X_i) = A(Y)$ and $f^*(X_i) < f^*(Y)$, then $M(A(X_1),\ldots,A(X_n)) = M(A(X_1),\ldots,A(X_{i-1}),A(Y),A(X_{i+1}),\ldots,A(X_n))$ and $\#X_i > \#Y$. So, by Lemma 4.3.1 we have that $A(\mathcal{M}_{A,f^*}(X_1,\ldots,X_n)) = A(\mathcal{M}_{A,f^*}(X_1,\ldots,X_{i-1},Y,X_{i+1},\ldots,X_n))$ and $f^*(\mathcal{M}_{A,f^*}(X_1,\ldots,X_n)) \leq f^*(\mathcal{M}_{A,f^*}(X_1,\ldots,X_{i-1},Y,X_{i+1},\ldots,X_n))$. □

Corollary 5.5.3 *Let a positive weighting vector $\omega = (w_1,w_2,\ldots w_n)$ and $OWA_\omega : [0,1]^n \to [0,1]$ be the OWA w.r.t. ω. The function $\mathcal{OWA}_\omega^{A,f^*} : \mathbb{H}^n \to \mathbb{H}$, defined by*

$$\mathcal{OWA}_\omega^{A,f^*}(X_1,\ldots,X_n) = A_{m\downarrow}^{(-1)}\left(OWA_\omega(A(X_1),\ldots,A(X_n))\right), \quad (28)$$

is an $\langle\mathbb{H},\preceq_A^{f^*}\rangle$-aggregation operator.

Proof: Since the OWA is strictly increasing, whenever ω is positive (see (BELIAKOV; PRADERA; CALVO, 2007, p.69)), then from Proposition 5.5.1, the corollary holds. □

Example 5.5.3 *Taking THFS and weighting vector in Example 5.5.1, by the action of $\mathcal{OWA}_\omega^{A,f}$ operator, we obtain the results:*

1. $\mathcal{OWA}_\omega^{A,f}(X_1) = A_{1\downarrow}^{(-1)}(0.1\cdot 0.4+0.2\cdot 0.7+0.3\cdot 1.0+0.4\cdot 1.0) = A_{1\downarrow}^{(-1)}(0.88) = \{0.88\} = \mathcal{OWA}_\omega^{Lex1}(X_1)$;
2. $\mathcal{OWA}_\omega^{A,f}(X_2) = \mathcal{OWA}_\omega^{Lex1}(X_2)$;
3. $\mathcal{OWA}_\omega^{A,f}(X_3) = \mathcal{OWA}_\omega^{Lex1}(X_3)$;
4. $\mathcal{OWA}_\omega^{A,f}(X_4) = \mathcal{OWA}_\omega^{Lex1}(X_4)$;
5. $\mathcal{OWA}_\omega^{A,f}(X_5) = A_{3\downarrow}^{(-1)}(0.1\cdot 0.344945+0.2\cdot 0.6715+0.3\cdot 1.0+0.4\cdot 1.0) = A_{3\downarrow}^{(-1)}(0.8687945) = \{0.8687945\}$;
6. $\mathcal{OWA}_\omega^{A,f}(X_6) = A_{3\downarrow}^{(-1)}(0.1\cdot 0.478+0.2\cdot 0.689+0.3\cdot 1+0.4\cdot 1) = A_{3\downarrow}^{(-1)}(0.8856) = \{0.85, 0.86\}$.

5.6 Chapter summary

In this section, the notion of some typical hesitant fuzzy connectives was presented, based on an admissible order $\preceq$ related to the poset $(\mathbb{H},\preceq)$. Then, considering an arbitrary admissible order $\preceq$ on $\mathbb{H}$, the concept of $\langle\mathbb{H},\preceq\rangle$-negation was presented. Also, we presented a simple method to generate $\langle\mathbb{H},\preceq\rangle$-negations w.r.t. three admissible orders, $\preceq_{Lex1}$-, $\preceq_{Lex2}$-, $\preceq_A^{f^*}$-orders.

Here, we also presented the definition of t-norms for typical hesitant fuzzy elements, considering admissible orders on $\mathbb{H}$.

In the following, the definition of $\langle \mathbb{H}, \preceq \rangle$-aggregation function was presented also considering an admissible $\langle \mathbb{H}, \preceq \rangle$-order. The concepts of $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-, $\langle \mathbb{H}, \preceq_{Lex2} \rangle$ and $\langle \mathbb{H}, \preceq_{A}^{f} \rangle$-OWA, were presented in this section as the ordered weighted average aggregations, considering the admissible $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-, $\langle \mathbb{H}, \preceq_{Lex2} \rangle$- and $\langle \mathbb{H}, \preceq_{A}^{f} \rangle$- orders, respectively.

6 TYPICAL HESITANT $\langle \mathbb{H}, \preceq \rangle$-IMPLICATIONS

In (BEDREGAL et al., 2014a), it was presented the extension of the notion of typical hesitant t-norms and in (BEDREGAL et al., 2014; SANTOS et al., 2014) the notion of typical hesitant negation for the partial orders considered in those papers. Here we introduce the notion of $\langle \mathbb{H}, \preceq \rangle$-implications, as typical hesitant fuzzy implications considering an admissible $\langle \mathbb{H}, \preceq \rangle$-order, discussing its main properties. From now on, we denote $\mathfrak{I}_{\langle \mathbb{H}, \preceq \rangle}$ as the set of all $\langle \mathbb{H}, \preceq \rangle$-implications.

6.1 Definition of $\langle \mathbb{H}, \preceq \rangle$-implications

The main fuzzy implications are extended to hesitant fuzzy implications, presenting the main properties, as antitonicity, isotonicity and corner conditions. In addition, to the extended examples of the main implications, we also studied methods of the conjugate that preserve these properties.

Definition 6.1.1 *A function $\mathcal{I} \colon \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is a typical hesitant fuzzy implication (THFI) w.r.t. $\langle \mathbb{H}, \preceq \rangle$-order, denotes as $\langle \mathbb{H}, \preceq \rangle$-implication, if for each $X, Y, Z \in \mathbb{H}$, the following properties are verified:*

($\mathcal{I}$1) If $X \preceq Y$, then $\mathcal{I}(Y, Z) \preceq \mathcal{I}(X, Z)$ (first place antitonicity);

($\mathcal{I}2$) If $Y \preceq Z$, then $\mathcal{I}(X, Y) \preceq \mathcal{I}(X, Z)$ (second place isotonicity);

($\mathcal{I}3$) $\mathcal{I}(\mathbf{0}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ (corner condition 1);

($\mathcal{I}4$) $\mathcal{I}(\mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ (corner condition 2); and

($\mathcal{I}5$) $\mathcal{I}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$ (corner condition 3).

The dual construction of a THFI $\mathcal{I} \colon \mathbb{H}^2 \to \mathbb{H}$ w.r.t. an $\langle \mathbb{H}, \preceq \rangle$-order, is a typical hesitant fuzzy coimplication (THFC), denoted as $\mathcal{J} \colon \mathbb{H}^2 \to \mathbb{H}$ verifying properties from $\mathcal{J}1$ to $\mathcal{J}5$.

Definition 6.1.2 *A function $\mathcal{J}\colon \mathbb{H}\times\mathbb{H}\to\mathbb{H}$ is a typical hesitant fuzzy coimplication (THFC) w.r.t. $\langle\mathbb{H},\preceq\rangle$-order, denotes as $\langle\mathbb{H},\preceq\rangle$-coimplication, if for each $X,Y,Z\in\mathbb{H}$, the following properties are verified:*

($\mathcal{J}1$) If $X\preceq Y$, then $\mathcal{J}(Y,Z)\preceq\mathcal{J}(X,Z)$ (first place antitonicity);

($\mathcal{J}2$) If $Y\preceq Z$, then $\mathcal{J}(X,Y)\preceq\mathcal{J}(X,Z)$ (second place isotonicity);

($\mathcal{J}3$) $\mathcal{J}(\mathbf{0}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}})=\mathbf{0}_{\mathbb{H}}$ (corner condition 1);

($\mathcal{J}4$) $\mathcal{J}(\mathbf{1}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}})=\mathbf{0}_{\mathbb{H}}$ (corner condition 2); and

($\mathcal{J}5$) $\mathcal{J}(\mathbf{0}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}})=\mathbf{1}_{\mathbb{H}}$ (corner condition 3).

6.2 Main properties and examples of $\langle\mathbb{H},\preceq\rangle$-implications

From these properties we can deduce that each $\langle\mathbb{H},\preceq\rangle$-(co)implication also satisfies two others.

Proposition 6.2.1 *Let $\mathcal{I}(\mathcal{J})\colon \mathbb{H}\times\mathbb{H}\to\mathbb{H}$ be an $\langle\mathbb{H},\preceq\rangle$-(co)implication. $\mathcal{I}(\mathcal{J})$ verifies the following properties:*

($\mathcal{I}6$) $\mathcal{I}(\mathbf{0}_{\mathbb{H}},Y)=\mathbf{1}_{\mathbb{H}}$ (left boundary);

($\mathcal{I}7$) $\mathcal{I}(X,\mathbf{1}_{\mathbb{H}})=\mathbf{1}_{\mathbb{H}}$ (right boundary).

($\mathcal{J}6$) $\mathcal{J}(\mathbf{1}_{\mathbb{H}},Y)=\mathbf{0}_{\mathbb{H}}$ (left boundary);

($\mathcal{J}7$) $\mathcal{J}(X,\mathbf{0}_{\mathbb{H}})=\mathbf{0}_{\mathbb{H}}$ (right boundary).

Proof: For all $X,Y\in\mathbb{H}$, the following holds:

($\mathcal{I}6$) By $\mathcal{I}2$, $\mathcal{I}(\mathbf{0}_{\mathbb{H}},Y)\succeq\mathcal{I}(\mathbf{0}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}})=\mathbf{1}_{\mathbb{H}}$, implying that $\mathcal{I}(\mathbf{0}_{\mathbb{H}},Y)=\mathbf{1}_{\mathbb{H}}$;

($\mathcal{I}7$) By $\mathcal{I}1$, $\mathcal{I}(X,\mathbf{1}_{\mathbb{H}})\succeq\mathcal{I}(\mathbf{1}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}})=\mathbf{1}_{\mathbb{H}}$, implying that $\mathcal{I}(X,\mathbf{1}_{\mathbb{H}})=\mathbf{1}_{\mathbb{H}}$.

Analogously, also can be prove for the dual construction. Therefore, Proposition 6.2.1 is verified. □

Extra properties of $\langle\mathbb{H},\preceq\rangle$-implications are reported in the following:

($\mathcal{I}8$) $\mathcal{I}(X,X)=\mathbf{1}_{\mathbb{H}}$ (identity principle);

($\mathcal{I}9$) $\mathcal{I}(X,\mathcal{I}(Y,Z))=\mathcal{I}(Y,\mathcal{I}(X,Z))$ (exchange principle);

($\mathcal{I}10$) $\mathcal{I}(X,Y)=\mathcal{I}(\mathcal{N}(Y),\mathcal{N}(X))$, if $\mathcal{N}$ is $\langle\mathbb{H},\preceq\rangle$-negation (contrapositive symmetry);

($\mathcal{I}11$) $X\preceq Y\Rightarrow\mathcal{I}(X,Y)=\mathbf{1}_{\mathbb{H}}$ (left-ordering property);

($\mathcal{I}12$) $\mathcal{I}(X,Y)=\mathbf{1}_{\mathbb{H}}\Rightarrow X\preceq Y$ (right ordering property).

Besides, for $\langle \mathbb{H}, \preceq \rangle$-coimplications, extra properties are reported in the following:

($\mathcal{J}8$) $\mathcal{J}(X,X) = \mathbf{0}_{\mathbb{H}}$ (normality condition);

($\mathcal{J}9$) $\mathcal{J}(X,\mathcal{J}(Y,Z)) = \mathcal{J}(Y,\mathcal{J}(X,Z))$ (exchange principle);

($\mathcal{J}10$) $\mathcal{J}(X,Y) = \mathcal{J}(\mathcal{N}(Y),\mathcal{N}(X))$, if $\mathcal{N}$ is $\langle \mathbb{H}, \preceq \rangle$-negation (contraposition);

($\mathcal{J}11$) $Y \preceq X \Rightarrow \mathcal{J}(X,Y) = \mathbf{0}_{\mathbb{H}}$ (left-ordering coimplication property);

($\mathcal{J}12$) $\mathcal{J}(X,Y) = \mathbf{0}_{\mathbb{H}} \Rightarrow Y \preceq X$ (right-ordering coimplication property).

Remark 6.2.1 $\left(\mathfrak{I}_{\langle \mathbb{H},\preceq \rangle}, \vee_{\langle \mathbb{H},\preceq \rangle}, \wedge_{\langle \mathbb{H},\preceq \rangle}, \mathcal{I}_\perp, \mathcal{I}_\top\right)$ *is a bounded lattice, where*

$$\mathcal{I}_\perp(X,Y) = \begin{cases} \mathbf{0}_{\mathbb{H}}, & \textit{if } X = \mathbf{1}_{\mathbb{H}} \textit{ and } Y = \mathbf{0}_{\mathbb{H}} \\ \mathbf{1}_{\mathbb{H}}, & \textit{otherwise;} \end{cases} \qquad \mathcal{I}_\top(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \textit{if } X = Y = \mathbf{1}_{\mathbb{H}} \textit{ or } X = Y = \mathbf{0}_{\mathbb{H}} \\ \mathbf{0}_{\mathbb{H}}, & \textit{otherwise;} \end{cases}$$

and the meet and joint-morphisms w.r.t $\langle \mathbb{H}, \preceq \rangle$ *are given as follows:*

$$\mathcal{I}_1 \wedge_{\langle \mathbb{H},\preceq \rangle} \mathcal{I}_2(X,Y) = \min_{\langle \mathbb{H},\preceq \rangle} \left(\mathcal{I}_1(X,Y), \mathcal{I}_2(X,Y)\right);$$
$$\mathcal{I}_1 \vee_{\langle \mathbb{H},\preceq \rangle} \mathcal{I}_2(X,Y) = \max_{\langle \mathbb{H},\preceq \rangle} \left(\mathcal{I}_1(X,Y), \mathcal{I}_2(X,Y)\right).$$

Example 6.2.1 *For each* $\preceq \in \mathfrak{O}_{\mathfrak{A}}$ *we obtain the Gödel and Rescher* $\langle \mathbb{H}, \preceq \rangle$*-implications:*

$$\mathcal{I}_{\mathcal{RS}}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \textit{if } X \preceq Y \\ \mathbf{0}_{\mathbb{H}}, & \textit{otherwise;} \end{cases} \qquad \mathcal{I}_{\mathcal{GD}}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \textit{if } X \preceq Y \\ Y, & \textit{otherwise.} \end{cases}$$

Independently from the $\langle \mathbb{H}, \preceq \rangle-$*order, we have the Weber hesitant fuzzy implication:*

$$\mathcal{I}_{\mathcal{WB}}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \textit{if } X \neq \mathbf{1}_{\mathbb{H}} \\ Y, & \textit{otherwise.} \end{cases}$$

So, $\langle \mathbb{H}, \preceq \rangle$*-implication refines results from* $\langle [0,1], \leq \rangle$*, meaning that* $\mathcal{I}_{\mathcal{RS}} \prec \mathcal{I}_{\mathcal{GD}} \prec \mathcal{I}_{\mathcal{WB}}$.

In the following, specific examples for the three admissible orders are presented.

Example 6.2.2 *Other examples related to* $\langle \mathbb{H}, \preceq_{Lex1} \rangle$*-implication are Goguen, Yager,* *Fodor and Łukasiewisc:*

$$\mathcal{I}_{\mathcal{GG}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex1} Y\\ \left\{\frac{Y^{(1)}}{X^{(1)}}\right\}, \text{ otherwise};\end{cases}$$

$$\mathcal{I}_{\mathcal{YG}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex1} Y\\ \left\{\left(Y^{(1)}\right)^{X^{(1)}}\right\}, \text{ otherwise};\end{cases}$$

$$\mathcal{I}_{\mathcal{FD}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex1} Y\\ \left\{\max\left(1-X^{(1)},Y^{(1)}\right)\right\}, \text{ otherwise};\end{cases}$$

$$\mathcal{I}_{\mathcal{LK}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex1} Y\\ \left\{1-X^{(1)}+Y^{(1)}\right\}, \text{ otherwise}.\end{cases}$$

In addition, $\langle\mathbb{H},\preceq_{Lex1}\rangle$ refines results from $\langle[0,1],\leq\rangle$, meaning that:

(1) $\mathcal{I}_{\mathcal{RS}} \prec_{Lex1} \mathcal{I}_{\mathcal{GD}} \prec_{Lex1} \mathcal{I}_{\mathcal{GG}} \prec_{Lex1} \mathcal{I}_{\mathcal{LK}} \prec_{Lex1} \mathcal{I}_{\mathcal{WB}}$*;*

(2) $\mathcal{I}_{\mathcal{RS}} \prec_{Lex1} \mathcal{I}_{\mathcal{GD}} \prec_{Lex1} \mathcal{I}_{\mathcal{FD}} \prec_{Lex1} \mathcal{I}_{\mathcal{LK}} \prec_{Lex1} \mathcal{I}_{\mathcal{WB}}$*; and*

(3) $\mathcal{I}_{\mathcal{YG}} \prec_{Lex1} \mathcal{I}_{\mathcal{LK}} \prec_{Lex1} \mathcal{I}_{\mathcal{WB}}$*.*

Example 6.2.3 *Analogous examples of $\langle\mathbb{H},\preceq_{Lex2}\rangle$-implication are reported:*

$$\mathcal{I}_{\mathcal{GG}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex2} Y\\ \left\{\frac{Y^{(\#Y)}}{X^{(\#X)}}\right\}, \text{ otherwise};\end{cases}$$

$$\mathcal{I}_{\mathcal{YG}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex2} Y\\ \left\{\left(Y^{(\#Y)}\right)^{X^{(\#X)}}\right\}, \text{ otherwise};\end{cases}$$

$$\mathcal{I}_{\mathcal{FD}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex2} Y\\ \left\{\max\left(1-X^{(\#X)},Y^{(\#Y)}\right)\right\}, \text{ otherwise};\end{cases}$$

$$\mathcal{I}_{\mathcal{LK}}(X,Y)=\begin{cases}\mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_{Lex2} Y\\ \left\{1-X^{(\#X)}+Y^{(\#Y)}\right\}, \text{ otherwise}.\end{cases}$$

The corresponding refinements are also obtained.

Example 6.2.4 *And, analogous $\langle \mathbb{H}, \preceq_A^f \rangle$-implications are then expressed below:*

$$\mathcal{I}_{\mathcal{GG}}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_A^f Y \\ A_{m\downarrow}^{(-1)}\left(\frac{A(Y)}{A(X)}\right), \text{ otherwise;} \end{cases}$$

$$\mathcal{I}_{\mathcal{YG}}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_A^f Y \\ A_{m\downarrow}^{(-1)}\left(A(Y)^{A(X)}\right), \text{ otherwise;} \end{cases}$$

$$\mathcal{I}_{\mathcal{FD}}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_A^f Y \\ A_{m\downarrow}^{(-1)}(\max(1 - A(X), A(Y)), \text{ otherwise;} \end{cases}$$

$$\mathcal{I}_{\mathcal{LK}}(X,Y) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } X \preceq_A^f Y \\ A_{m\downarrow}^{(-1)}(1 - A(X) + A(Y)), \text{ otherwise;} \end{cases}$$

if $m = \max(\#X, \#Y)$ and $A_{m\downarrow}^{(-1)} \colon [0,1] \to \mathbb{H}$ as given in Theorem 5.2.1.

6.3 Natural $\langle \mathbb{H}, \preceq \rangle$-negations obtained from $\langle \mathbb{H}, \preceq \rangle$-implications

In (SANTOS et al., 2014) the notion of typical hesitant fuzzy negations (THFN) uses Xu-Xia-partial order and in (BEDREGAL et al., 2014) another order on $\mathbb{H}$ was discussed for the THFN. Here, a method to obtain Natural $\langle \mathbb{H}, \preceq \rangle$-negations obtained from $\langle \mathbb{H}, \preceq \rangle$-implications is presented.

Proposition 6.3.1 *Let $\preceq$ be an admissible partial order on $\mathbb{H}$ and let the function $\mathcal{I}(\mathcal{J})$ be an $\langle \mathbb{H}, \preceq \rangle$-(co)implication. Then, the function $\mathcal{N}_{\mathcal{I}(\mathcal{J})} : \mathbb{H} \to \mathbb{H}$, defined by*

($\mathcal{I}13$) $\mathcal{N}_{\mathcal{I}}(X) = \mathcal{I}(X, \mathbf{0}_{\mathbb{H}})$, *($\mathcal{J}13$) $\mathcal{N}_{\mathcal{J}}(X) = \mathcal{J}(X, \mathbf{1}_{\mathbb{H}})$,*

is an $\langle \mathbb{H}, \preceq \rangle$-negation, called as the natural typical hesitant fuzzy negation of $\mathcal{I}(\mathcal{J})$, or the natural $\langle \mathbb{H}, \preceq \rangle$-negation $\mathcal{N}_{\mathcal{I}(\mathcal{J})}$.

Proof: From the corner conditions, $\mathcal{N}_{\mathcal{I}}(\mathbf{0}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$ and $\mathcal{N}_{\mathcal{I}}(\mathbf{1}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$. If $X \preceq Y$, then by $(\mathcal{I}1)$, it holds that $\mathcal{N}_{\mathcal{I}}(Y) = \mathcal{I}(Y, \mathbf{0}_{\mathbb{H}}) \preceq \mathcal{I}(X, \mathbf{0}_{\mathbb{H}}) = \mathcal{N}_{\mathcal{I}}(X)$. Analogously, it can be prove for the dual construction. Therefore, Preposition 6.3.1 holds. □

Example 6.3.1 *Let $\preceq \in \mathfrak{O}_{\mathfrak{A}}$. See examples of natural $\langle \mathbb{H}, \preceq \rangle$-negations:*

(i) For each $\preceq \in \mathfrak{O}_{\mathfrak{A}}$: $\mathcal{N}_{\mathcal{I}_{\mathcal{RS}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{GD}}} = \mathcal{N}_{\perp}$ and $\mathcal{N}_{\mathcal{I}_{\mathcal{WB}}} = \mathcal{N}_{\top}$.

(ii) For $\langle \mathbb{H}, \preceq_{Lex1} \rangle-$order, $\mathcal{N}_{\mathcal{I}_{\mathcal{LK}}}(X) = \mathcal{N}_{\mathcal{I}_{\mathcal{FD}}}(X) = \{1 - X^{(1)}\}$, $\forall X \in \mathbb{H}$. Then, by Example 6.2.2, we have the following comparisons:

(a) $\mathcal{N}_{\perp} = \mathcal{N}_{\mathcal{I}_{\mathcal{RS}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{GD}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{GG}}} \prec_{Lex1} \mathcal{N}_{\mathcal{I}_{\mathcal{LK}}} \prec_{Lex1} \mathcal{N}_{\mathcal{I}_{\mathcal{WB}}} = \mathcal{N}_{\top}$;

(b) $\mathcal{N}_{\perp} = \mathcal{N}_{\mathcal{I}_{\mathcal{RS}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{GD}}} \prec_{Lex1} \mathcal{N}_{\mathcal{I}_{\mathcal{FD}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{LK}}} \prec_{Lex1} \mathcal{N}_{\mathcal{I}_{\mathcal{WB}}} = \mathcal{N}_{\top}$; and

(c) $\mathcal{N}_\perp = \mathcal{N}_{\mathcal{I}_{\mathcal{YG}}} \prec_{Lex1} \mathcal{N}_{\mathcal{I}_{\mathcal{LK}}} \prec_{Lex1} \mathcal{N}_{\mathcal{I}_{\mathcal{WB}}} = \mathcal{N}_\top$*.*

(iii) For $\langle \mathbb{H}, \preceq_{Lex2}\rangle$*−order,* $\forall X \in \mathbb{H}$*, we obtain the following expression* $\mathcal{N}_{\mathcal{I}_{\mathcal{LK}}}(X) = \mathcal{N}_{\mathcal{I}_{\mathcal{FD}}}(X) = \{1 - X^{(\#X)}\}$*. Then, by Example 6.2.3, we have analogous comparisons as presented in (a), (b) and (c).*

(iii) For $\langle \mathbb{H}, \preceq^f_A\rangle$*−order,* $\forall X \in \mathbb{H}$*, we have that*

$$\mathcal{N}_{\mathcal{I}_{\mathcal{LK}}}(X) = \mathcal{N}_{\mathcal{I}_{\mathcal{FD}}}(X) = \mathcal{N}_{\mathcal{I}_{\mathcal{YG}}}(X) = A^{(-1)}_{\#X\downarrow}(0.\overline{9} - A(X)).$$

In particular, when A *is the function in Proposition 4.3.1, then* $\mathcal{N}_{\mathcal{I}_{\mathcal{LK}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{FD}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{YG}}} = \mathcal{N}_{\mathcal{S}_{A,f}}$*. Then, by Example 6.2.4, analogous comparisons as (a), (b) and (c) can also be obtained together with the following:* $\mathcal{N}_\perp = \mathcal{N}_{\mathcal{I}_{\mathcal{YG}}} = \mathcal{N}_{\mathcal{I}_{\mathcal{LK}}} \prec^f_A \mathcal{N}_{\mathcal{I}_{\mathcal{WB}}} = \mathcal{N}_\top$*.*

Proposition 6.3.2 *Let* $\mathcal{I}$ *be an* $\langle \mathbb{H}, \preceq\rangle$*-implication verifying properties* $(\mathcal{I}8)$*,* $(\mathcal{I}9)$ *and* $(\mathcal{I}12)$*. Then, the following holds:*

(i) $X \preceq \mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(X))$ *for each* $X \in \mathbb{H}$*;*

(ii) $\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(X))) = \mathcal{N}_\mathcal{I}(X)$ *for each* $X \in \mathbb{H}$*;*

(iii) $\mathcal{N}_\mathcal{I}$ *is strong if and only if* $\mathcal{N}_\mathcal{I}$ *is surjective.*

Proof: Let $\mathcal{I}$ be an $\langle \mathbb{H}, \preceq\rangle$-implication satisfying $(\mathcal{I}8)$, $(\mathcal{I}9)$ and $(\mathcal{I}12)$. So:

(i) By $(\mathcal{I}8)$ and $(\mathcal{I}9)$, $\mathcal{I}(X, \mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H})) = \mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathcal{I}(X, \mathbf{0}_\mathbb{H})) = \mathbf{1}_\mathbb{H}$. Therefore, by $(\mathcal{I}12)$, $X \preceq \mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H})) = \mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(X))$.

(ii) Since $\mathcal{N}_\mathcal{I}$ is decreasing, from the previous item one can conclude that $\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(X))) \preceq \mathcal{N}_\mathcal{I}(X)$, for each $X \in \mathbb{H}$. On the other hand, for all $X \in \mathbb{H}$, by $(\mathcal{I}9)$ and $(\mathcal{I}8)$, we obtain the following equation: $\mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathcal{I}(\mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H})) = \mathcal{I}(\mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H}), \mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H})) = \mathbf{1}_\mathbb{H}$. So, by $(\mathcal{I}12)$, it holds that $\mathcal{I}(X, \mathbf{0}_\mathbb{H}) \preceq \mathcal{I}(\mathcal{I}(\mathcal{I}(X, \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H}), \mathbf{0}_\mathbb{H})$, i.e., $\mathcal{N}_\mathcal{I}(X) \preceq \mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(X)))$, for $X \in \mathbb{H}$. Therefore, $\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(X))) = \mathcal{N}_\mathcal{I}(X)$.

(iii) $(\Rightarrow)$ Let $X \in \mathbb{H}$ and $Y = \mathcal{N}_\mathcal{I}(X)$. Since $\mathcal{N}_\mathcal{I}$ is strong, $\mathcal{N}_\mathcal{I}(Y) = X$.

$(\Leftarrow)$ Let $X \in \mathbb{H}$. Since $\mathcal{N}_\mathcal{I}$ is surjective, there is $Y \in \mathbb{H}$ such that $\mathcal{N}_\mathcal{I}(Y) = X$. So, by the previous item, $\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(X)) = \mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(\mathcal{N}_\mathcal{I}(Y))) = \mathcal{N}_\mathcal{I}(Y) = X$.

Concluding, this yields the desired result and Proposition 6.3.2 is verified. □

6.4 Generation of $\langle \mathbb{H}, \preceq \rangle$-implications from fuzzy implications

The order-preserving method to generate $\langle \mathbb{H}, \preceq^f_A \rangle$-implications from fuzzy implications is presented, based on representable $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-implications.

6.4.1 $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-implications preserving main fuzzy implications

Common properties of implications are preserved by $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-implications.

Theorem 6.4.1 *Let $I_1, I_2, \cdots, I_n(J_1, J_2, \cdots, J_n) : [0,1]^2 \to [0,1]$ be functions verifying $Ii(Ji)$, for some $i \in \mathbb{N}_8$, $I10(J10)$ for a fuzzy negation N, and $I11(J11)$. The function $\widetilde{I_1, \ldots, I_n}(\widetilde{J_1, \ldots, J_n}) : \mathbb{H}^2 \to \mathbb{H}$ defined by*

$$\widetilde{I_1, \ldots, I_n}(X, Y) = \{I_k(X^{(1)}, Y^{(1)}) : k \in \mathbb{N}_n\}, \tag{29}$$

$$\widetilde{J_1, \ldots, J_n}(X, Y) = \{J_k(X^{(1)}, Y^{(1)}) : k \in \mathbb{N}_n\}, \tag{30}$$

verifies $\mathcal{I}i(\mathcal{J}i)$, for $i \in \mathbb{N}_8$, for $i = 11$ and for $i = 10$, w.r.t. $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-negation $\widetilde{N}$.

Proof: Let $I_1, \cdots, I_n : [0,1]^2 \to [0,1]$ be functions verifying properties $I1$-$I8$ and $I10$-$I11$.

($\mathcal{I}1$) By Eq. (17), if $X_1 \preceq_{Lex1} X_2$ then $X_1^{(1)} \leq X_2^{(1)}$. So, by Eq. (29) and I1 we have that $\widetilde{I_1, \ldots, I_n}(X_1, Y) = \{I_k(X_1^{(1)}, Y^{(1)}) : k \in \mathbb{N}_n\} \succeq_{Lex1} \{I_k(X_2^{(1)}, Y^{(1)}) : k \in \mathbb{N}_n\} = \widetilde{I_1, \ldots, I_n}(X_2, Y)$.

($\mathcal{I}2$) By Eq. (17), $Y_1 \preceq_{Lex1} Y_2$ implies that $Y_1^{(1)} \leq Y_2^{(1)}$. So, by Eq. (29) and I2, $\widetilde{I_1, \ldots, I_n}(X, Y_1) = \{I_k(X^{(1)}, Y_1^{(1)}) : k \in \mathbb{N}_n\} \preceq_{Lex1} \{I_k(X^{(1)}, Y_2^{(1)}) \colon k \in \mathbb{N}_n\} = \widetilde{I_1, \ldots, I_n}(X, Y_2)$.

($\mathcal{I}3$) By Eq.(29) and $I3$, $\widetilde{I_1, \ldots, I_n}(\mathbf{0}_\mathbb{H}, \mathbf{0}_\mathbb{H}) = \{\mathcal{I}_k(0,0) \colon k \in \mathbb{N}_n\} = \{1\} = \mathbf{1}_\mathbb{H}$.

($\mathcal{I}4$) By Eq.(29) and $I4$, $\widetilde{I_1, \ldots, I_n}(\mathbf{1}_\mathbb{H}, \mathbf{1}_\mathbb{H}) = \{\mathcal{I}_k(1,1) \colon k \in \mathbb{N}_n\} = \{1\} = \mathbf{1}_\mathbb{H}$.

($\mathcal{I}5$) By Eq.(29) and $I5$, $\widetilde{I_1, \ldots, I_n}(\mathbf{1}_\mathbb{H}, \mathbf{0}_\mathbb{H}) = \{\mathcal{I}_k(1,0) \colon k \in \mathbb{N}_n\} = \{0\} = \mathbf{0}_\mathbb{H}$.

($\mathcal{I}6$) By Eq.(29) and $I6$, $\widetilde{I_1, \ldots, I_n}(\mathbf{0}_\mathbb{H}, Y) = \{I_k(0, Y^{(1)}) \colon k \in \mathbb{N}_n\} = \{1\} = \mathbf{1}_\mathbb{H}$.

($\mathcal{I}7$) By Eq.(29) and $I7$, $\widetilde{I_1, \ldots, I_n}(X, \mathbf{1}_\mathbb{H}) = \{I_k(X^{(1)}, 1) \colon k \in \mathbb{N}_n\} = \{1\} = \mathbf{1}_\mathbb{H}$.

($\mathcal{I}8$) By Eq.(29) and $I8$, $\widetilde{I_1, \ldots, I_n}(X, X) = \{I_k(X^{(1)}, X^{(1)}) \colon k \in \mathbb{N}_n\} = \{1\} = \mathbf{1}_\mathbb{H}$.

($\mathcal{I}10$) If I_k verifies $I10$ w.r.t. an N fuzzy negation. Hence, by Eq.(29) we obtain: $\widetilde{I_1, \ldots, I_n}\left(\widetilde{N}(Y), \widetilde{N}(X)\right) = \{I_k(N(Y^{(1)}), N(X^{(1)})) \colon k \in \mathbb{N}_n\} = \{I_k(X^{(1)}, Y^{(1)}) \colon k \in \mathbb{N}_n\} = \widetilde{I_1, \ldots, I_n}(X, Y)$, where $\widetilde{N}$ is the $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-negation generated from N, in Prop. 5.2.1.

($\mathcal{I}11$) By Eq.(17), $X \preceq_{Lex1} Y$ implies that $X^{(1)} \leq Y^{(1)}$. Thus, by Eq.(29) and $I11$, $\widetilde{I_1, \ldots, I_n}(X, Y) = \{I_k(X^{(1)}, Y^{(1)})\colon\ k \in \mathbb{N}_n\} = \{1\} = \mathbf{1}_{\mathbb{H}}$.

Analogously, we can prove for the dual construction, the representable coimplication $\widetilde{J_1, \ldots, J_n}$. Therefore Theorem 6.4.1 holds. □

Corollary 6.4.1 *If* $I_1, I_2, \cdots, I_n$ *(*$J_1, J_2, \cdots, J_n$*):* $[0,1]^2 \rightarrow [0,1]$ *are fuzzy (co)implications, then the function* $\widetilde{I_1, \ldots, I_n}, \widetilde{J_1, \ldots, J_n}\colon \mathbb{H}^2 \rightarrow \mathbb{H}$*, defined by Eq. (29) and Eq. (30) is an* $\langle \mathbb{H}, \preceq_{Lex1} \rangle$*-(co)implication.*

Proof: It follows straightforward from Theorem 6.4.1. □

In the Theorem 6.4.1 converse construction, the analysis of the conditions result from the proposed properties of representable $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-implications that can be restricted to fuzzy implication functions based on diagonal elements of THFS.

Theorem 6.4.2 *Let* $\mathcal{I}(\mathcal{J})$ *be an* $\langle \mathbb{H}, \preceq_{Lex1} \rangle$*-(co)implication verifying properties* $\mathcal{I}k(\mathcal{J}k)$*,* $k \in \mathbb{N}_{12}$*, except* $k = 9$*, and considering* $\mathcal{I}10(\mathcal{J}10)$ *w.r.t. an* $\langle \mathbb{H}, \preceq_{Lex1} \rangle$*-negation* $\widetilde{N}$ *generated from a fuzzy negation* N *as in Proposition 5.2.1. The function* $I(J)\colon [0,1]^2 \rightarrow [0,1]$ *given as* $I(x, y) = (\mathcal{I}(\{x\}, \{y\}))^{(1)}, (J(x, y) = (\mathcal{J}(\{x\}, \{y\}))^{(1)})$*, verifies properties* $Ik(Jk)$*,* $k \in \mathbb{N}_{12}$*, except* $k = 9$*, and considering* $I10(J10)$ *w.r.t. the fuzzy negation* N*.*

Proof: Straightforward. □

Proposition 6.4.1 $\langle \mathbb{H}, \preceq_{Lex1} \rangle$*-negation w.r.t. the* $\langle \mathbb{H}, \preceq_{Lex1} \rangle$*-(co)implication* $\widetilde{I_1, \ldots, I_n}$ *(*$\widetilde{J_1, \ldots, J_n}$*) is, respectively, given as*

$$\mathcal{N}_{\widetilde{I_1, \ldots, I_n}}(X) = \{N_{I_k}(X^{(1)})\colon k \in \mathbb{N}_n\}, \forall X \in \mathbb{H}, \tag{31}$$

$$\mathcal{N}_{\widetilde{J_1, \ldots, J_n}}(X) = \{N_{J_k}(X^{(1)})\colon k \in \mathbb{N}_n\}, \forall X \in \mathbb{H}. \tag{32}$$

Proof: For all $X \in \mathbb{H}$, we have that $\mathcal{N}_{\widetilde{I_1, \ldots, I_n}}(X) = \widetilde{I_1, \ldots, I_n}(X, \mathbf{0}_{\mathbb{H}}) = \{I_k(X^{(1)}, 0)\colon k \in \mathbb{N}_n\} = \{N_{I_k}(X^{(1)})\colon k \in \mathbb{N}_n\}$. In addition, the boundary conditions are also verified:

($\mathcal{N}1$) $\mathcal{N}_{\widetilde{I_1, \ldots, I_n}}(\mathbf{0}_{\mathbb{H}}) = \{N_{I_k}(0)\colon k \in \mathbb{N}_n\} = \mathbf{1}_{\mathbb{H}}$; and $\mathcal{N}_{\widetilde{I_1, \ldots, I_n}}(\mathbf{1}_{\mathbb{H}}) = \{N_{I_k}(1)\colon k \in \mathbb{N}_n\} = \mathbf{0}_{\mathbb{H}}$;

($\mathcal{N}2$) And, when $X_1 \preceq_{Lex1} X_2$, then by $\mathcal{I}1$ the following holds: $\mathcal{N}_{\widetilde{I_1, \ldots, I_n}}(X_1) = \widetilde{I_1, \ldots, I_n}(X_1, \mathbf{0}_{\mathbb{H}}) \succeq_{Lex1} \widetilde{I_1, \ldots, I_n}(X_2, \mathbf{0}_{\mathbb{H}}) = \mathcal{N}_{\widetilde{I_1, \ldots, I_n}}(X_2)$.

The dual construction is analogously achieved. Therefore, Proposition 6.4.1 holds. □

6.4.2 $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-implication as a structure-preserving properties

See in the following analogous results from Proposition 6.4.3 which can be extended to representable $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-implications.

Theorem 6.4.3 *Let $I_1, I_2, \cdots, I_n(J_1, J_2, \cdots, J_n) : [0,1]^2 \to [0,1]$ be functions verifying a property from $Ii(Ji)$, for some $i \in \mathbb{N}_8$, $I10(J10)$ w.r.t. a negation N, and $I11(J11)$. The function $\widetilde{\widetilde{I_1, \ldots,}} I_n(\widetilde{\widetilde{J_1, \ldots,}} J_n) : \mathbb{H}^2 \to \mathbb{H}$ defined by*

$$\widetilde{\widetilde{I_1, \ldots,}} I_n(X, Y) = \{I_k(X^{(\#X)}, Y^{(\#Y)}) : k \in \mathbb{N}_n\}, \tag{33}$$
$$\widetilde{\widetilde{J_1, \ldots,}} J_n(X, Y) = \{J_k(X^{(\#X)}, Y^{(\#Y)}) : k \in \mathbb{N}_n\}, \tag{34}$$

verifies property $\mathcal{I}i(\mathcal{J}i)$, for $i \in \mathbb{N}_8$, and also for $\mathcal{I}11(\mathcal{J}11)$ and $\mathcal{I}10(\mathcal{J}10)$ w.r.t. to the $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-natural negation $\widetilde{\widetilde{N}}$.

Proof: Analogously done in Theorem 6.4.1. □

Corollary 6.4.2 *If $I_1, I_2, \cdots, I_n(J_1, J_2, \cdots, J_n) : [0,1]^2 \to [0,1]$ are fuzzy (co)implication functions then the function $\widetilde{\widetilde{I_1, \ldots,}} I_n(\widetilde{\widetilde{J_1, \ldots,}} J_n) : \mathbb{H}^2 \to \mathbb{H}$ defined by Eq.(33) (and Eq.(34)) is an $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-(co)implication.*

Proof: Follows directly from Theorem 6.4.3. □

Theorem 6.4.4 *Let $\mathcal{I}(\mathcal{J})$ be an $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-(co)implication verifying properties $\mathcal{I}k(\mathcal{J}k)$, $k \in \mathbb{N}_{12}$, except $k = 9$, and considering $\mathcal{I}10(\mathcal{J}10)$ w.r.t. an $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-negation $\widetilde{\widetilde{N}}$ generated from a negation N as Prop. 5.2.1. The function $I(J) : [0,1]^2 \to [0,1]$, given as $I(x,y) = (\mathcal{I}(\{x\},\{y\}))^{(m)}, (J(x,y) = (\mathcal{J}(\{x\},\{y\}))^{(m)})$, where $m = \#\mathcal{I}(\{x\},\{y\}), (m = \#\mathcal{J}(\{x\},\{y\}))$, verifies $Ik(Jk)$, except $k = 9$ considering $I10(J10)$ w.r.t. the fuzzy negation N.*

Proof: Straightforward. □

Proposition 6.4.2 *From Theorem 6.4.4, the natural $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-negation is given as*

$$\mathcal{N}_{\widetilde{\widetilde{I_1,\ldots,I_n}}}(X) = \{N_{I_k}(X^{(\#X)}) \colon k \in \mathbb{N}_n\}, \forall X \in \mathbb{H}, \tag{35}$$
$$\mathcal{N}_{\widetilde{\widetilde{J_1,\ldots,J_n}}}(X) = \{N_{J_k}(X^{(\#X)}) \colon k \in \mathbb{N}_n\}, \forall X \in \mathbb{H}. \tag{36}$$

Proof: For all $X \in \mathbb{H}$, $\mathcal{N}_{\widetilde{\widetilde{I_1,\ldots,I_n}}}(X) = \widetilde{\widetilde{I_1, \ldots,}} I_n(X, \mathbf{0}_{\mathbb{H}}) = \{I_k(X^{(\#X)}, 0) \colon k \in \mathbb{N}_n\} = \{N_{I_k}(X^{(\#X)}) \colon k \in \mathbb{N}_n\}$. In addition, the boundary conditions are also verified:

($\mathcal{N}1$) $\mathcal{N}_{\widetilde{\widetilde{I_1,\ldots,I_n}}}(\mathbf{0}_{\mathbb{H}}) = \{N_{I_k}(0) \colon k \in \mathbb{N}_n\} = \mathbf{1}_{\mathbb{H}}$; and $\mathcal{N}_{\widetilde{\widetilde{I_1,\ldots,I_n}}}(\mathbf{1}_{\mathbb{H}}) = \{N_{I_k}(1) \colon k \in \mathbb{N}_n\} = \mathbf{0}_{\mathbb{H}}$;

$(\mathcal{N}2)$ If $X_1 \preceq_{Lex2} X_2$, $\mathcal{N}_{\widetilde{\widetilde{I_1,\ldots,I_n}}}(X_1) = \widetilde{\widetilde{I_1,\ldots,I_n}}(X_1, \mathbf{0}_{\mathbb{H}}) \succeq_{Lex2} \widetilde{\widetilde{I_1,\ldots,I_n}}(X_2, \mathbf{0}_{\mathbb{H}}) = \mathcal{N}_{\widetilde{\widetilde{I_1,\ldots,I_n}}}(X_2)$.

The dual construction is analogously achieved. Therefore, Proposition 6.4.2 holds. □

6.4.3 $\langle \mathbb{H}, \preceq_A^f \rangle-$implication as a structure-preserving properties

The generation of $\langle \mathbb{H}, \preceq_A^f \rangle$-implications from fuzzy implication functions are detailed, if $A\colon \mathbb{H} \to [0,1]$ satisfies the conditions of Theorem 4.3.2 and $f^* : \mathbb{H} \to \mathbb{R}$ verifies IC.

Theorem 6.4.5 *Let $\mathcal{N}_{A,f^*}$ be an $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-negation. Let $I(J) : [0,1]^2 \to [0,1]$ be a function verifying the property $Ii(Ji)$, for some $i \in \mathbb{N}_8$ and also $I10(J10)$ and $I11(J11)$, and in case $i \in \{10\}$, w.r.t. a negation N. The function $\mathcal{I}_{A,f^*}(\mathcal{J}_{A,f^*})\colon \mathbb{H}^2 \to \mathbb{H}$ given by*

$$\mathcal{I}_{A,f^*}(X,Y) = A_{n\downarrow}^{(-1)}(I(A(X), A(Y))), \quad (37)$$

$$\mathcal{J}_{A,f^*}(X,Y) = A_{n\downarrow}^{(-1)}(J(A(X), A(Y))), \quad (38)$$

when $n = \max\{\#X, \#Y\}$, verifies $\mathcal{I}i$ ($\mathcal{J}i$) related to $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-order, and for $i = 10$, w.r.t. the $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-negation $\mathcal{N}_{A,f^*}$. In addition, if $I(J)$ verifies $I12$ then $\mathcal{I}_{A,f^*}$ ($\mathcal{J}_{A,f^*}$) verifies

$(\mathcal{I}12a)$ *If $\mathcal{I}_{A,f^*}(X,Y)$ then $A(X) \leq A(Y)$, $\forall X, Y \in \mathbb{H}$.*

Proof: Let $I : [0,1]^2 \to [0,1]$ be a function verifying Ik, $k \in \mathbb{N}_{12}$. The following holds:

$(\mathcal{I}1)$ If $X_1 \preceq_A^{f^*} X_2$, then either $A(X_1) < A(X_2)$ or $A(X_1) = A(X_2)$ and $f^*(X_1) \leq f^*(X_2)$. By $I1$, $I(A(X_1), A(Y)) \geq I(A(X_2), A(Y))$ and it holds that

$$\begin{aligned}\mathcal{I}_{A,f^*}(X_1,Y) &= A_{n\downarrow}^{(-1)}(I(A(X_1), A(Y))), \text{ by Eq.(37)}\\ &\succeq_A^{f^*} A_{n\downarrow}^{(-1)}(I(A(X_2), A(Y))) = \mathcal{I}_{A,f^*}(X_2,Y), \text{ by Lemma 4.3.1 and Eq.(19).}\end{aligned}$$

Otherwise, if $A(X_1) = A(X_2)$ and $f^*(X_1) \leq f^*(X_2)$ then $A(\mathcal{I}_{A,f^*}(X_1,Y)) = A(\mathcal{I}_{A,f^*}(X_2,Y))$ and $\#X_2 \leq \#X_1$ and by Lemma 4.3.1 and $I1$, $f^*(\mathcal{I}_{A,f^*}(X_2,Y)) \leq f^*(\mathcal{I}_{A,f^*}(X_1,Y))$. So, $\mathcal{I}_{A,f^*}(X_2,Y)) \preceq_A^{f^*} \mathcal{I}_{A,n\downarrow}(X_1,Y)$.

$(\mathcal{I}2)$ If $Y_1 \preceq_A^{f^*} Y_2$ then either $A(Y_1) < A(Y_2)$ or $A(Y_1) = A(Y_2)$ and $f^*(Y_1) \leq f^*(Y_2)$. By $I2$, $I(A(X), A(Y_1)) \leq I(A(X), A(Y_2))$ implying in the following result

$$\begin{aligned}\mathcal{I}_{A,f^*}(X,Y_1) &= A_{n\downarrow}^{(-1)}(I(A(X), A(Y_1))) \text{ by Eq.(37)}\\ &\preceq_A^{f^*} A_{n\downarrow}^{(-1)}(I(A(X), A(Y_2))) = \mathcal{I}_{A,f^*}(X,Y_2), \text{ by Lemma 4.3.1 and Eq.(19).}\end{aligned}$$

Otherwise, if $A(Y_1) = A(Y_2)$ and $f^*(Y_1) \leq f^*(Y_2)$ then $A(\mathcal{I}_{A,f^*}(X,Y_1)) = A(\mathcal{I}_{A,f^*}(X,Y_2))$ and $\#Y_2 \leq \#Y_1$ and by Lemma 4.3.1 and $I2$, $f^*(\mathcal{I}_{A,f^*}(X,Y_1)) \leq f^*(\mathcal{I}_{A,f^*}(X,Y_2))$. So, $\mathcal{I}_{A,f^*}(X,Y_1)) \preceq_A^{f^*} \mathcal{I}_{A,f^*}(X,Y_2)$.

($\mathcal{I}3$) $\mathcal{I}_{A,f^*}(\mathbf{0}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = A^{(-1)}_{1\downarrow}(I(A(\mathbf{0}_{\mathbb{H}}), A(\mathbf{0}_{\mathbb{H}}))) = A^{(-1)}_{n\downarrow}(I(0,0)) = A^{(-1)}_{1\downarrow}(1) = \mathbf{1}_{\mathbb{H}}$.

($\mathcal{I}4$) $\mathcal{I}_{A,f^*}(\mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}) = A^{(-1)}_{1\downarrow}(I(A(\mathbf{1}_{\mathbb{H}}), A(\mathbf{1}_{\mathbb{H}}))) = A^{(-1)}_{1\downarrow}(I(1,1)) = A^{(-1)}_{1\downarrow}(1) = \mathbf{1}_{\mathbb{H}}$.

($\mathcal{I}5$) $\mathcal{I}_{A,f^*}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = A^{-1}_{1\downarrow}(I(A(\mathbf{1}_{\mathbb{H}}), A(\mathbf{0}_{\mathbb{H}}))) = A^{-1}_{1\downarrow}(I(1,0)) = A^{-1}_{1\downarrow}(0) = \mathbf{0}_{\mathbb{H}}$.

($\mathcal{I}6$) Follows from Proposition 6.2.1.

($\mathcal{I}7$) Follows from Proposition 6.2.1.

($\mathcal{I}8$) By Eq.(37) and Lemma 4.3.1, $\mathcal{I}_{A,f^*}(X,X) = A^{(-1)}_{n\downarrow}(I(A(X), A(X))) = A^{(-1)}_{n\downarrow}(1) = \mathbf{1}_{\mathbb{H}}$.

($\mathcal{I}10$) Let $\mathcal{N}_{A,f^*} : \mathbb{H} \to \mathbb{H}$ be the $\langle \mathbb{H}, \preceq^{f^*}_A \rangle$-negation generated from a fuzzy negation N as in Theorem 5.2.1. Then, if I satisfies $I12$ for N, then the following holds:

$$\begin{aligned}\mathcal{I}_{A,f^*}(\mathcal{N}_{A,f^*}(Y), \mathcal{N}_{A,f^*}(X)) &= A^{-1}_{n\downarrow}(I(A(\mathcal{N}_{A,f^*}(Y), A(\mathcal{N}_{A,f^*}(X))))), \text{ by Eq. (37);}\\ &= A^{(-1)}_{n\downarrow}(I(A(A^{(-1)}_{n\downarrow}(N(A(Y)))), A(A^{(-1)}_{n\downarrow}(N(A(X)))))) \text{ by Eq.(21);}\\ &= A^{(-1)}_{n\downarrow}(I(A(X), A(Y))) = \mathcal{I}_{A,f^*}(X,Y), \text{ by Lemma 4.3.1 and } I12.\end{aligned}$$

($\mathcal{I}11$) If $X \preceq^{f^*}_A Y$ then $A(X) \leq A(Y)$. Then by $I10$, $I(A(X), A(Y)) = 1$. Hence, by Lemma 4.3.1, $\mathcal{I}_{A,f^*}(X,Y) = A^{(-1)}_{n\downarrow}(I(A(X), A(Y))) = A^{(-1)}_{n\downarrow}(1) = \mathbf{1}_{\mathbb{H}}$.

($\mathcal{I}12a$) If $\mathcal{I}_{A,f^*}(X,Y) = \mathbf{1}_{\mathbb{H}}$ it means that $A^{-1}_{n\downarrow}(I(A(X), A(Y))) = \mathbf{1}_{\mathbb{H}}$. Then, by Lemma 4.3.1, $I(A(X), A(Y)) = 1$ and by $I12$ $A(X) \leq A(Y)$.

Analogously, we can prove for the dual construction $\mathcal{J}_{A,f^*}(X,Y)$. Therefore, Theorem 6.4.5 is verified. □

In the following theorem, we present a method to obtain $\langle [0,1], \leq \rangle$-implications from $\langle \mathbb{H}, \preceq^{f^*}_A \rangle$-implications based on the family of functions $A^{(-1)}_n$, defined in Lemma 4.3.1.

Theorem 6.4.6 *Let $\mathcal{I}(\mathcal{J})$ be an $\langle \mathbb{H}, \preceq^{f^*}_A \rangle$-function given by Eq.(37) (and Eq. (38)) verifying properties $\mathcal{I}i(\mathcal{J}i)$, for $i \in \mathbb{N}_{12}$, except $i = 9$, and for the case of $\mathcal{I}10(\mathcal{J}10)$, by considering the $\langle \mathbb{H}, \preceq^{f^*}_A \rangle$-negation as $\mathcal{N}_{A,f^*}$. For each $n \in \mathbb{N}^+$, $I_{A,n}(J_{A,n}) : [0,1]^2 \to [0,1]$ given as*

$$I_{A,n}(x,y) = A(\mathcal{I}(A^{(-1)}_{n\downarrow}(x), A^{(-1)}_{n\downarrow}(y))), \tag{39}$$

$$J_{A,n}(x,y) = A(\mathcal{J}(A^{(-1)}_{n\downarrow}(x), A^{(-1)}_{n\downarrow}(y))), \tag{40}$$

verifies property $Ii(Ji)$, except $i = 9$, and for $I10(J10)$ by considering the fuzzy negation as N.

Proof: Respecting the above conditions, we have the next following results:

(I1) If $x_1 \leq x_2$, by Lemma 4.3.1, $A_{n\downarrow}^{(-1)}(x_1) \leq A_{n\downarrow}^{(-1)}(x_2)$. By $\mathcal{I}1$, $\mathcal{I}(A_{n\downarrow}^{(-1)}(x_1), A_{n\downarrow}^{(-1)}(y)) \succeq_A^f \mathcal{I}(A_{n\downarrow}^{(-1)}(x_2), A_{n\downarrow}^{(-1)}(y))$. By Eq. (39) and since A is increasing w.r.t. $\preceq_A^f$, $I_{A,n}(x_1, y) = A(\mathcal{I}(A_{n\downarrow}^{-1}(x_1), A_n^{-1}(y))) \geq A(\mathcal{I}(A_{n\downarrow}^{-1}(x_2), A_n^{-1}(y))) = I_{A,n}(x_2, y)$.

(I2) If $y_1 \leq y_2$, by Lemma 4.3.1, $A_{n\downarrow}^{(-1)}(y_1) \leq A_{n\downarrow}^{(-1)}(y_2)$, and then by $\mathcal{I}2$, $\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y_1)) \preceq_A^f \mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y_2))$. So, by Eq. (39) and because A is increasing w.r.t. $\preceq_A^f$, $I_{A,n}(x, y_1) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y_1))) \leq A(\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y_2))) = I_{A,n}(x, y_2)$.

(I3) By $\mathcal{I}3$, $I_{A,n}(0,0) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(0), A_{n\downarrow}^{(-1)}(0))) = A(\mathcal{I}(\mathbf{0}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}})) = A(\mathbf{1}_{\mathbb{H}}) = 1$.

(I4) By $\mathcal{I}4$, $I_{A,n}(1,1) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(1), A_{n\downarrow}^{(-1)}(1))) = A(\mathcal{I}(\mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}})) = A(\mathbf{1}_{\mathbb{H}}) = 1$.

(I5) By $\mathcal{I}5$, $I_{A,n}(1,0) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(1), A_{n\downarrow}^{(-1)}(0))) = A(\mathcal{I}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}})) = A(\mathbf{0}_{\mathbb{H}}) = 0$.

(I6) By $\mathcal{I}6$, $I_{A,n}(0,y) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(0), A_{n\downarrow}^{(-1)}(y))) = A(\mathcal{I}(\mathbf{0}_{\mathbb{H}}, A_{n\downarrow}^{-1}(y))) = A(\mathbf{1}_{\mathbb{H}}) = 1$.

(I7) By $\mathcal{I}7$, $I_{A,n}(x,1) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(1))) = A(\mathcal{I}(A_{n\downarrow}^{-1}(x), \mathbf{0}_{\mathbb{H}})) = A(\mathbf{1}_{\mathbb{H}}) = 1$.

(I8) By $\mathcal{I}8$, $I_{A,n}(x,x) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(x))) = A(\mathbf{1}_{\mathbb{H}}) = 1$.

(I10) Let $\mathcal{I}$ verifying $\mathcal{I}12$ w.r.t. the $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-negation $\mathcal{N}_{A,f^*}$ generated from a fuzzy negation N. Then, by $\mathcal{I}12$, and property IC and Eq. (39), $I_{A,n}(x,y) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y))) = A(\mathcal{I}(\mathcal{N}_{A,f^*}(A_{n\downarrow}^{(-1)}(y)), \mathcal{N}_{A,f^*}(A_{n\downarrow}^{(-1)}(x)))) = A(\mathcal{I}(A_{n\downarrow}^{(-1)}(N(A(A_{n\downarrow}^{(-1)}(y))), A_{n\downarrow}^{(1)}(N(A(A_{n\downarrow}^{(-1)}(x)))) = I_{A,n}(N(y), N(x))$.

(I11) If $x \leq y$, by Lemma 4.3.1, $A_{n\downarrow}^{-1}(x) \preceq_A^{f*} A_{n\downarrow}^{-1}(y)$, implying by $\mathcal{I}10$, that $\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y)) = \mathbf{1}_{\mathbb{H}}$. Consequently $A(\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y))) = 1$ and $I_{A,n}(x,y) = 1$,

(I12) If $I_{A,n}(x,y) = 1$ then, by Eq. (39), $A(\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y))) = 1$ and, by Lemma 4.3.1, $\mathcal{I}(A_{n\downarrow}^{(-1)}(x), A_{n\downarrow}^{(-1)}(y)) = \mathbf{1}_{\mathbb{H}}$. So, by $\mathcal{I}12$, $A_{n\downarrow}^{(-1)}(x) \preceq_A^{f^*} A_{n\downarrow}^{(-1)}(y)$ and then, $A(A_{n\downarrow}^{(-1)}(x)) \leq A(A_{n\downarrow}^{(-1)}(y))$. Hence, by Lemma 4.3.1, $x \leq y$.

Analogously, we can prove for the dual construction $J_{A,n}(X,Y)$. Therefore, Theorem 6.4.6 holds. □

Proposition 6.4.3 *Let $N_I(N_J) : [0,1] \to [0,1]$ be the natural $\langle [0,1], \leq \rangle$-negation of a fuzzy (co)implication $I(J)$ and $\mathcal{I}_{A,f^*}(\mathcal{J}_{A,f^*}) : \mathbb{H}^2 \to \mathbb{H}$ be the $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-(co)implication generated from $I(J)$ in Theorem 6.4.5. Then, the natural $\langle \mathbb{H}, \preceq_A^{f^*} \rangle$-negation of $\mathcal{I}_{A,f^*}(\mathcal{J}_{A,f^*})$ is the function $\mathcal{N}_{\mathcal{I}_{A,f^*}}(\mathcal{N}_{\mathcal{J}_{A,f^*}}) : \mathbb{H} \to \mathbb{H}$ given as follows*

$$\mathcal{N}_{\mathcal{I}_{A,f^*}}(X) = A_{n\downarrow}^{-1}(N_I(A(X))) \ \forall X \in \mathbb{H}; \tag{41}$$

$$\mathcal{N}_{\mathcal{J}_{A,n}}(X) = A^{-1}_{n\downarrow}(N_J(A(X)))\ \forall X \in \mathbb{H}, \tag{42}$$

when $n = \#X$*.*

Proof: By Eq. (37), we have that $\mathcal{N}_{\mathcal{I}_{A,f^*}}(X) = \mathcal{I}_{A,f^*}(X, \mathbf{0}_{\mathbb{H}}) = A^{-1}_{n\downarrow}(\mathcal{I}(A(X), A(\mathbf{0}_{\mathbb{H}}))) = A^{-1}_{n\downarrow}(I_{A,n}(A(X), A(0))) = A^{-1}_{n\downarrow}(N_I(A(X)))$. Besides, by Prop. 6.3.1, $\mathcal{N}_{\mathcal{I}_{A,f^*}}$ is an $\langle \mathbb{H}, \preceq^{f^*}_A \rangle$-negation. Analogously, we can prove for the dual construction. Therefore, Proposition 6.4.3 holds. □

6.5 Chapter summary

In this chapter, we introduced the notion of $\langle \mathbb{H}, \preceq \rangle$-implications, as typical hesitant fuzzy implications considering an admissible $\langle \mathbb{H}, \preceq \rangle$-order, discussing its main properties. The main fuzzy implications were extended to hesitant fuzzy implications, presenting the main properties, as antitonicity, isotonicity and corner conditions. In addition, to the extended examples of the main implications, we also studied methods of the conjugate that preserve these properties.

Besides, the concept of $\langle \mathbb{H}, \preceq \rangle$-negation was presented based on an arbitrary admissible $\langle \mathbb{H}, \preceq \rangle$-order, and an order-preserving method to obtain $\langle \mathbb{H}, \preceq \rangle$-implications from fuzzy implications is given, based on representable $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-implications. The generation of $\langle \mathbb{H}, \prec^f_A \rangle$-implications from fuzzy implication functions are also detailed.

7 PROPOSED STRATEGY FOR SOLVING A ME-MCDM PROBLEM

Straight to the applied soft computing, this section describes the proposed strategy to solve a ME-MCDM (Multi Expert-Multi Criteria Decision Making) problem considering the $\langle \mathbb{H}, \preceq \rangle$-orders introduced in previous sections. Firstly, see Fig. 1, the diagram summarizing the main classes and their attributes, discussed in this article and which are involved in the next algorithmic proposal. In hesitant fuzzy environments, in order to solve a process of alternatives sorting analysis in the class of ME-MCDM problems, we can explore the admissibility, linearity and refinement related to $\langle \mathbb{H}, \preceq \rangle$-orders, as the ordered-structure supporting the main attributes in the classes identifying fuzzy connectives (negations, aggregation and implication functions).

The application described in (WEN; ZHAO; XU, 2019, Example 1) is considered as a case-study for the selection of a support-software, enabling its validation and comparison. Note that we present detailed calculation descriptions regarding the $\langle \mathbb{H}, \preceq_A^f \rangle$-order. The other ones can be analogously done.

7.1 $\langle \mathbb{H}, \preceq \rangle$- Algorithm-solution for ME-MCDM problem

In order to help the user in problems involving decision making based on many experts and multiple criterion. See, e.g., in the selection of an software systems available in the market nowadays, a data processing company aims to clarify differences of such systems (CHEN; XU; XIA, 2013). In this case study, let $\mathcal{A} = \{A_1, A_2, \ldots, A_{n_2}\}$ ($\#\mathcal{A} = n_2$) be the set of software alternatives and $\mathcal{T} = \{T_1, \ldots, T_{n_1}\}$ ($\#\mathcal{T} = n_1$) be the set of the selected criterion, including a team of experts $\mathcal{E} = \{E_1, \ldots E_n\}$ ($n = \#\mathcal{E}$). So, the proposal methodology to solve this ME-MCDM problem is described in the following steps:

Step 1 Define n, n_1 and n_2, reporting the number of experts, the number of attributes and the number of alternatives in the modelled application, respectively.

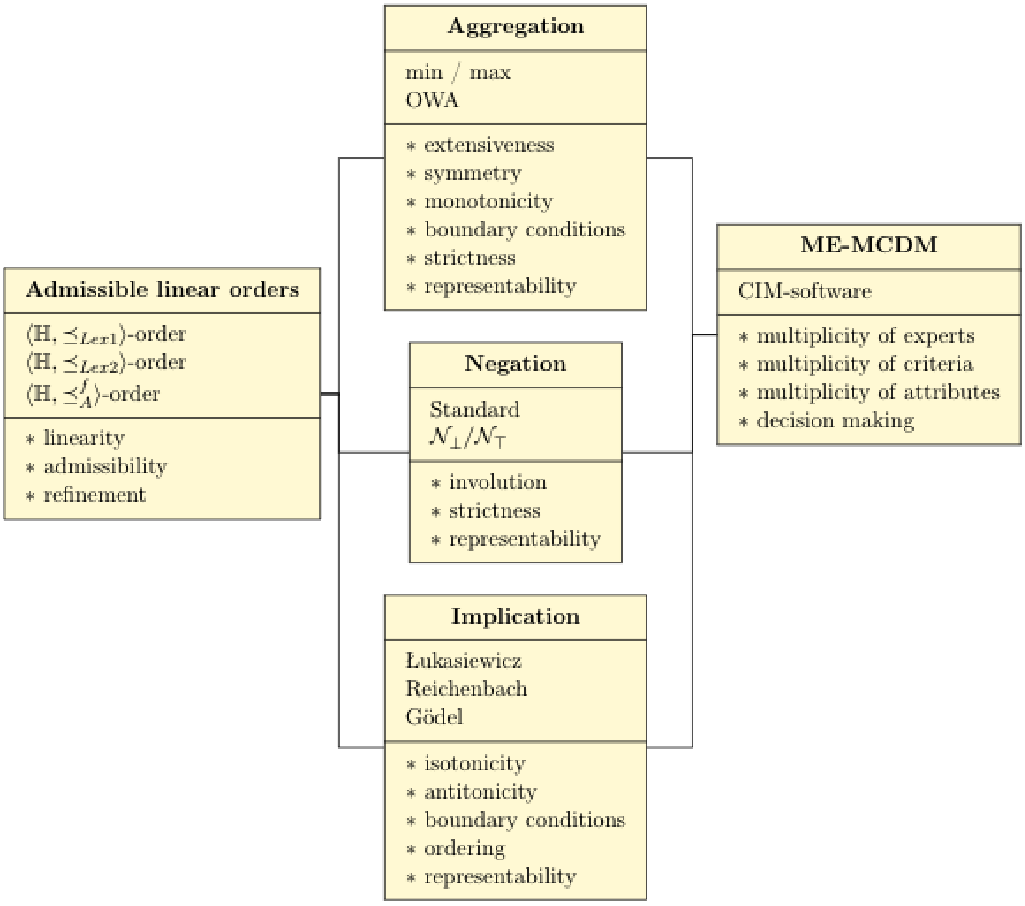

Figure 1 – Main classes and their attributes involved in the proposed algorithm

Step 2 Select the $\langle \mathbb{H}, \preceq\rangle$-order and the corresponding $\langle \mathbb{H}, \preceq\rangle$-implication and aggregation;

Step 3 Determine each THFS $\mathbf{x_{ik}}$, for $i \in \mathbb{N}_{n_2}$ and $k \in \mathbb{N}_{n_1}$, associating to each pair alternative-criterion the expert opinions including their corresponding aggregation values, resulting on $n_2 \times n_1$ tabular data structure;

Step 4 Determine each THFS $\mathbf{z}_{ij}(\mathbf{x}_k) = \mathcal{I}(\mathbf{x}_{ik}, \mathbf{x}_{jk})$ obtained by the action of an $\langle \mathbb{H}, \preceq\rangle$-implication in the THFS of the previous step, which results on sets of at most n_1 THFS as the components of the $n_2 \times n_2$ tabular data, $\forall i, j \in \mathbb{N}_{n_2}$ and $\forall k \in \mathbb{N}_{n_1}$. As examples, for the proposed three admissible order approaches:

(i) $\langle \mathbb{H}, \preceq_A^f\rangle$-order, for A and f from Theorem 4.3.2, and $\langle \mathbb{H}, \preceq_A^f\rangle$-implication $\mathcal{I}_{LK}$ as given in Example 6.2.4, then we have that

$$\mathbf{z}_{ij}(\mathbf{x}_k) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{ik}, \mathbf{x}_{jk}) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } \mathbf{x}_{ik} \preceq_A^f \mathbf{x}_{jk}; \\ A_{n\downarrow}^{(-1)}(1 - A(\mathbf{x}_{ik}) + A(\mathbf{x}_{jk})), \text{ otherwise.} \end{cases}$$

(ii) $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-order, and an $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-implication $\mathcal{I}_{LK}$ from Example 6.2.2, then

$$\mathbf{z}_{ij}(\mathbf{x}_k) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{ik}, \mathbf{x}_{jk}) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } \mathbf{x}_{ik} \preceq_{Lex1} \mathbf{x}_{jk} \\ \{1 - \mathbf{x}_{ik} + \mathbf{x}_{jk} : i \in \mathbb{N}_n, n = \min(\#X, \#Y)\}, \text{otherwise.} \end{cases}$$

(iii) $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-order, and an $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-implication $\mathcal{I}_{LK}$ from Example 6.2.3, then

$$\mathbf{z}_{ij}(\mathbf{x}_k) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{ik}, \mathbf{x}_{jk}) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } \mathbf{x}_{ik} \preceq_{Lex2} \mathbf{x}_{jk} \\ \{1 - \mathbf{x}_{ik} + \mathbf{x}_{jk} : i \in \mathbb{N}_n, n = \min(\#X, \#Y)\}, \text{otherwise.} \end{cases}$$

Step 5 Apply the selected $\langle \mathbb{H}, \preceq \rangle$-order and take an increasing ordering of all the resulting k-components $\mathbf{z}_{ij}(\mathbf{x}_k)$, for all $k \in \mathbb{N}_{n_1}$, related to each $i, j \in \mathbb{N}_{n_2}$;

Step 6 Electing the best alternative, based on the selected $\langle \mathbb{H}, \preceq \rangle$-aggregation performed over the resulting k-components $\mathbf{z}_{ij}(\mathbf{x}_k)$, for all $k \in \mathbb{N}_{n_1}$, related to each $k \in \mathbb{N}_{n1}$, and $i, j \in \mathbb{N}_{n_2}$, such that $i \leq j$ and considering the following cases:

(i) $\langle \mathbb{H}, \preceq_A^f \rangle$-OWA in Eq. (28): $\mathcal{O}\left(\mathbf{z}_{ij}(\mathbf{x}_k)_{k \in \mathbb{N}_{n1}}\right)_{i,j \in \mathbb{N}_{n2}} \preceq_A^f \mathcal{O}\left(\mathbf{z}_{ji}(\mathbf{x}_k)_{k \in \mathbb{N}_{n1}}\right)_{j,i \in \mathbb{N}_{n2}}$;

(ii) $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-OWA in Eq. (24):$\mathcal{O}\left(\mathbf{z}_{ij}(\mathbf{x}_k)_{k \in \mathbb{N}_{n1}}\right)_{i,j \in \mathbb{N}_{n2}} \preceq_{Lex1} \mathcal{O}\left(\mathbf{z}_{ji}(\mathbf{x}_k)_{k \in \mathbb{N}_{n1}}\right)_{j,i \in \mathbb{N}_{n2}}$;

(iii) $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-OWA in Eq.(26):$\mathcal{O}\left(\mathbf{z}_{ij}(\mathbf{x}_k)_{k \in \mathbb{N}_{n1}}\right)_{i,j \in \mathbb{N}_{n2}} \preceq_{Lex2} \mathcal{O}\left(\mathbf{z}_{ji}(\mathbf{x}_k)_{k \in \mathbb{N}_{n1}}\right)_{j,i \in \mathbb{N}_{n2}}$.

Since, for all other cases ($i > j$) the comparison has already been done, we have that $\mathbf{A_i}$ is a better alternative than $\mathbf{A_j}$, denoted as $\mathbf{A_i} \sqsupset \mathbf{A_j}$.

7.2 Case-study: Solving a ME-MCDM problem in a CIM-application

This section extends the application described in (WEN; ZHAO; XU, 2019, Example 1) from HFS to THFS, in order to solve the ME-MCDM problem under several alternatives in the selection of a CIM (Computer-Integrated Manufacturing) software, enabling further comparison.

The data of the CIM ME-MCDM problem were extracted from the application presented in (WEN; ZHAO; XU, 2019), meaning that in order to help the user in the selection of software systems which are available in the market, a data processing company aims to clarify differences of such systems.

Step 1: In this case study, we are considering the following data:

1. Let $\mathcal{A} = \{\mathbf{A}1, \ldots, \mathbf{A}7\}$ ($n_2 = 7$) be the set of software alternatives;

2. Let X be the set of the selected attributes: (i) functionality (x_1); (ii) usability (x_2); (iii) portability (x_3); and (iv) maturity (x_4) ($n_1 = 4$); and

3. It also includes a team of three experts ($n = 3$).

Step 2: This step is based on the following data:

1. We are considering the three admissible orders: $\langle \mathbb{H}, \preceq_A^f \rangle$, for A and f being the functions in Example 4.3.1, including $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders, extending the usual restrictive partial $\langle \mathbb{H}, \leq_{RH} \rangle$-order;

2. In addition, let $\mathcal{I}_{\mathcal{LK}} \colon (\mathbb{H})^2 \to \mathbb{H}$ be the corresponding representable hesitant fuzzy implication function obtained from Łukasiewicz fuzzy implication w.r.t. the previous three admissible orders selected, as presented in Examples 6.2.2, 6.2.3 and 6.2.4, respectively;

3. And, finally, we take the extended aggregation function: $\min : \bigcup_{n=1}^{\infty} [0,1]^4 \to [0,1]$.

Step 3: Based on the data extracted from CIM-application in (WEN; ZHAO; XU, 2019):

1. See Table 4, expressing THFS $(\mathbf{x}_{ik})_{i \in \mathbb{N}_7, k \in \mathbb{N}_4}$, whose THFE are describing the data evaluations of all three decision makers, as values between 0 and 1, for each data pair w.r.t. alternative-attributes of the proposed application;

2. It is placed in additional lines, the values obtained by the action of the selected aggregation operators performed over each THFS, and reported as: $A(\mathbf{x}_{ik}),\ \min(\mathbf{x}_{ik})$ and $\max(\mathbf{x}_{ik}), \forall i \in \mathbb{N}_7, k \in \mathbb{N}_;$.

3. And, summarizing, in the 5^{th} column, the comparison among them is reported, based on the selected admissible orders. Observe that the ordering related to $\langle \mathbb{H}, \preceq_A^f \rangle$-order coincides with comparisons obtained from $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders.

Step 4: Now, the results obtained by applying the $\mathcal{I}_{\mathcal{LK}}$ implications generated w.r.t. $\langle \mathbb{H}, \preceq_A^f \rangle$-, $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-, $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders, as reported in Table 5, Table 6 and Table 7, respectively. And corresponding $\langle \mathbb{H}, \preceq_A^f \rangle$-operator calculations are presented.

1. Firstly, we take the $\langle \mathbb{H}, \preceq_A^f \rangle$-order, for some functions A and f from Theorem 4.3.2, and the $\langle \mathbb{H}, \preceq_A^f \rangle$-implication $\mathcal{I}_{\mathcal{LK}}$ as given in Example 6.2.4. For $n_2 = 7$, $n_1 = 4$ and $n = 3$, all components of the 7×7 tabular data consider 4 THFS with at most 3 THFE, expressed for all $i, j \in \mathbb{N}_7$ and $k \in \mathbb{N}_4$ as follows:

$$\mathbf{z}_{ij}(\mathbf{x}_k) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{ik}, \mathbf{x}_{jk}) = \begin{cases} \mathbf{1}_{\mathbb{H}}, & \text{if } \mathbf{x}_{ik} \preceq_A^f \mathbf{x}_{jk}; \\ A_{n\downarrow}^{(-1)}(1 - A(\mathbf{x}_{ik}) + A(\mathbf{x}_{jk})), & \text{otherwise}; \end{cases} \quad (43)$$

2. In the sequence, the action of the $\mathcal{I}_{\mathcal{LK}}$ implication, given by Eq.(43), over pair-line components (i, j) and (j, i) of Table 4 results on data reported in Table 5,

Table 4 – Information related to THFS and admissible $\langle \mathbb{H}, \preceq \rangle$-orders.

		$k=1$	$k=2$	$k=3$	$k=4$	**Ordering**
	THSF $\mathbf{x_{1k}}$	$\{0.80, 0.85, 0.95\}$	$\{0.70, 0.75, 0.80\}$	$\{0.65, 0.80\}$	$\{0.30, 0.35\}$	
A1	$A(\mathbf{x_{1k}})$	0.889055	0.778050	0.685000	0.330500	$\mathbf{x_{11} \succeq x_{12} \succeq x_{13} \succeq x_{14}}$
	$\min(\mathbf{x_{1k}})$	0.80	0.70	0.65	0.30	
	$\max(\mathbf{x_{1k}})$	0.95	0.80	0.80	0.35	
	THFS $\mathbf{x_{2k}}$	$\{0.85, 0.90\}$	$\{0.60, 0.70, 0.80\}$	$\{0.20\}$	$\{0.15\}$	
A2	$A(\mathbf{x_{2k}})$	0.895000	0.678000	0.200000	0.150000	$\mathbf{x_{21} \succeq x_{22} \succeq x_{23} \succeq x_{24}}$
	$\min(\mathbf{x_{2k}})$	0.85	0.60	0.20	0.15	
	$\max(\mathbf{x_{2k}})$	0.90	0.80	0.20	0.15	
	THFS $\mathbf{x_{3k}}$	$\{0.20, 0.30, 0.40\}$	$\{0.40, 0.50\}$	$\{0.90, 1.00\}$	$\{0.45, 0.50, 0.65\}$	
A3	$A(\mathbf{x_{3k}})$	0.234	0.45	0.990909	0.456505	$\mathbf{x_{33} \succeq x_{34} \succeq x_{32} \succeq x_{31}}$
	$\min(\mathbf{x_{3k}})$	0.20	0.40	0.90	0.45	
	$\max(\mathbf{x_{3k}})$	0.40	0.50	1.00	0.65	
	THFS $\mathbf{x_{4k}}$	$\{0.80, 0.95, 1.00\}$	$\{0.10, 0.15, 0.20\}$	$\{0.20, 0.30\}$	$\{0.60, 0.70, 0.80\}$	
A4	$A(\mathbf{x_{4k}})$	0.899059	0.112050	0.230000	0.678000	$\mathbf{x_{41} \succeq x_{44} \succeq x_{43} \succeq x_{42}}$
	$\min(\mathbf{x_{4k}})$	0.80	0.10	0.20	0.60	
	$\max(\mathbf{x_{4k}})$	1.00	0.20	0.30	0.80	
	THFS $\mathbf{x_{5k}}$	$\{0.35, 0.40, 0.50\}$	$\{0.70, 0.90, 1.00\}$	$\{0.40\}$	$\{0.20, 0.30, 0.35\}$	
A5	$A(\mathbf{x_{5k}})$	0.345500	0.799009	0.400000	0.233005	$\mathbf{x_{52} \succeq x_{53} \succeq x_{51} \succeq x_{54}}$
	$\min(\mathbf{x_{5k}})$	0.35	0.70	0.40	0.20	
	$\max(\mathbf{x_{5k}})$	0.50	1.00	0.40	0.35	
	THFS $\mathbf{x_{6k}}$	$\{0.50, 0.60, 0.70\}$	$\{0.80, 0.90\}$	$\{0.40, 0.60\}$	$\{0.10, 0.20\}$	
A6	$A(\mathbf{x_{6k}})$	0.567000	0.890000	0.460000	0.120000	$\mathbf{x_{62} \succeq x_{61} \succeq x_{63} \succeq x_{64}}$
	$\min(\mathbf{x_{6k}})$	0.50	0.80	0.40	0.10	
	$\max(\mathbf{x_{6k}})$	0.70	0.90	0.60	0.20	
	THFS $\mathbf{x_{7i}}$	$\{0.80, 1.00\}$	$\{0.15, 0.20, 0.35\}$	$\{0.10, 0.20\}$	$\{0.70, 0.85\}$	
A7	$A(\mathbf{x_{7k}})$	0.8909	0.123505	0.120000	0.780500	$\mathbf{x_{71} \succeq x_{74} \succeq x_{72} \succeq x_{73}}$
	$\min(\mathbf{x_{7k}})$	0.80	0.15	0.10	0.70	
	$\max(\mathbf{x_{7k}})$	1.00	0.35	0.20	0.85	

according to $\mathbf{z}_{ij}(\mathbf{x}_k)$ calculations, for all $i, j \in \mathbb{N}_7$ and $k \in \mathbb{N}_4$, reporting its related aggregation.

In order to get a better understanding of such reported results, an example presents calculations which are restricted to components $\mathbf{z}_{13}(\mathbf{x}_i)$ and $\mathbf{z}_{31}(\mathbf{x}_i)$, related to the action of the implication function $\mathcal{I}_{\mathcal{LK}}$ over THFS from the first line and the third column ($i = 1; j = 3$) and the converse ($i = 3; j = 1$), for all $k \in \mathbb{N}_4$, respectively. The results from Eq. (43) are expressed in the following:

(I) Based on data from Table 4, first line and third column ($i = 1; j = 3$), for all $k \in \mathbb{N}_4$, see the achieved THFS given by the next expression:

$$\mathbf{z_{13}}(\mathbf{x}_k) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{1k}, \mathbf{x}_{3k}) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } \mathbf{x}_{1k} \preceq_A^f \mathbf{x}_{3k} \\ A_{n\downarrow}^{(-1)}(1 - A(\mathbf{x}_{1k}) + A(\mathbf{x}_{3k})), \text{ otherwise}; \end{cases} \quad (44)$$

Table 5 – Partial Overview Analysis of the CIM-Application using $\langle \mathbb{H}, \preceq_A^f \rangle$-order.

	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$A7$
A1	1;1;1;1	1; 0.89995; 0.5195; 0.8195	0.344945;0.6715; 1;1	1;0.334; 0.345;1	0.456445;1; 0.715;0.902505	0.677945;1; 0.775;0.7895	1;0.345455; 0.435;1
	{1};{1};{1};{1}	{1};{0.895,0.99}; {0.5195};{0.8195}	{0.39,0.44,0.45}; {0.615,0.79};{1};{1}	{1};{0.3,0.4}; {0.3,0.4,0.5};{1}	{0.44,0.54,0.65};{1}; {0.715};{0.902505}	{0.69,0.74,0.75}; {1};{0.75,0.8,0.9}	{1};{0.34,0.45,0.55}; {0.435};{1}
	1	0.885835	0.8688845	0.8424	0.859396	0.8596445	0.8215445
A2	1;1;1;1	1;1;1;1	0.339; 0.772;1;1	1;0.43405;1;1	0.4505;1;1;1	0.672;1;1;0.97	0.9959;0.445505; 0.92;1
	{1};{1};{1};{1}	{1};{1};{1};{1}	{0.339}; {0.772}; {1};{1}	{1};{0.43405}; {1};{1}	{0.4,0.55};{1}; {1}{1}	{0.62,0.7};{1}; {1};{0.97}	{0.95,0.99};{0.445505}; {0.92};{1}
	1	1	0.8883	0.943405	0.94505	0.9612	0.9273205
A3	1;1; 0.6896;0.873995	1;1; 0.6891;0.874095	1;1;1;1	1;0.66205; 0.2391;1	1;1;0.4091; 0.7765	1;1;0.4691; 0.663495	1;0.673505; 0.1291;1
	{1};{1};{0.69,0.86}; {0.873995}	{1};{1}; {0.69,0.81};{0.874095}	{1};{1};{1};{1}	{1};{0.66205}; {0.29,0.31};{1}	{1};{1}; {0.4091};{0.7765}	{1};{1};{0.49,0.61}; {0.639,0.645}	{1};{0.63,0.755}; {0.19,0.21};{1}
	0.944209	0.859609	1	0.85632	0.89621	0.879609	0.847611
A4	0.989996;1; 1;0.6525	0.995941;1; 0.97;0.472	0.334941;1; 1;0.778505	1;1;1;1	0.446441;1; 1;0.555005	0.667941;1; 1;0.442	0.991841;1; 0.89;1
	{0.989996};{1};{1}; {0.6525}	{0.995941};{1};{0.97}; {0.42,0.7}	{0.344,0.391};{1}; {1};{0.778505}	{1};{1};{1};{1}	{0.446441};{1}; {1}{0.555005}	{0.674,0.691};{1}; {1};{0.442}	{0.914,0.981};{1}; {0.8,0.9};{1}
	0.9632492	0.9399823	0.8891951	1	0.8556451	0.8777882	0.9873682
A5	1;0.979041; 1;1	1;0.878991; 0.8;0.916995	0.8885;0.650991 1;1	1;0.313041; 0.83;1	1;1;1;1	1;1; 1;0.886995	1;0.324496; 0.72;1
	{1};{0.979041}; {1};{1}	{1};{0.878991}; {0.8};{0.916995}	{0.8885};{0.650991}; {1};{1}	{1};{0.313041}; {0.83};{1}	{1};{1};{1};{1}	{1};{1};{1}; {0.869,0.895}	{1};{0.324496}; {0.72};{1}
	0.9979041	0.9308967	0.9427991	0.8973041	1	0.9886995	0.8764496
A6	1;0.88805; 1;1	1;0.788; 0.74;1	0.676;0.56; 1;1	1;0.22205; 0.77;1	0.7785;0.909009; 0.94;1	1;1;1;1	1;0.233505; 0.66;1
	{1};{0.88805}; {1};{1}	{1};{0.78,0.8}; {0.74};{1}	{0.66,0.7}; {0.5,0.6}; {1};{1}	{1};{0.22205}; {0.77};{1}	{0.7785};{0.909009}; {1};{0.94}	{1};{1};{1};{1}	{1}{0.230,0.355}; {0.66};{1}
	0.988805	0.9316	0.8894	0.876205	0.9416518	1	0.8553505
A7	0.998155;1; 1;0.55	1;1;1; 0.3695	0.3431;1; 1;0.676005	1;0.988545; 1;0.8975	0.4546;1; 1;0.452505	0.6761;1; 1;0.3395	1;1;1;1
	{0.998155};{1}; {1};{0.55}	{1};{1}; {1};{0.39,0.65}	{0.33,0.41};{1}; {1};{0.660,0.705}	{1};{0.988545}; {1};{0.87,0.95}	{0.44,0.56};{1}; {1};{0.420,0.555}	{0.66,0.71};{1}; {1}{0.3395}	{1};{1};{1};{1}
	0.954631	0.93695	0.869511	0.987459	0.8361705	0.86917	1

(a) Since $A(\{0.80, 0.85, 0.95\}) = 0.889055 \geq 0.234 = A(\{0.20, 0.30, 0.40\})$, we obtain that $\mathbf{x}_{11} = \{0.80, 0.85, 0.95\} \succeq^f_A \mathbf{x}_{31} = \{0.20, 0.30, 0.40\}$, meaning that $\mathbf{z_{13}}(\mathbf{x}_1) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{11}, \mathbf{x}_{31}) = \mathcal{I}_{\mathcal{LK}}(\{0.80, 0.85, 0.95\}, \{0.20, 0.30, 0.40\}) = A_3^{(-1)}(1 - 0.889055 + 0.234)$. So, $\mathbf{z_{13}}(\mathbf{x}_1) = A_3^{(-1)}(0, 344945) = \{0.39, 0.44, 0.45\}$. Concluding, $A(\mathbf{z_{13}}(\mathbf{x}_1)) = 0.344945.$

(b) Since $A(\{0.70, 0.75, 0.80\}\}) = 0.778050 \geq 0.45 = A(\{0.40, 0.50\})$, then we have as result that $\mathbf{x}_{12} = \{0.70, 0.75, 0.80\} \succeq^f_A \{0.40, 0.50\} = \mathbf{x}_{32}$. If $n = \min(3, 2) = 2$, then $\mathbf{z_{13}}(\mathbf{x}_2) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{12}, \mathbf{x}_{32}) = \mathcal{I}_{\mathcal{LK}}(\{0.70, 0.75, 0.80\}, \{0.40, 0.50\}) = A_2^{(-1)}(1 - 0.778050 + 0.45) = A_2^{(-1)}(0, 6715) = \{0.61, 0.75\}$. So, $A(\mathbf{z_{13}}(\mathbf{x}_2)) = 0.6715.$

(c) Since $A(\{0.65, 0.80\}) = 0.685 \leq 0.990989 = A(\{0.90, 1.00\})$, then we have that: $\mathbf{x}_{13} = \{0.65, 0.80\} \preceq^f_A \{0.90, 1.00\} = \mathbf{x}_{33}$, implying the results: $\mathbf{z}_{13}(\mathbf{x}_3) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{13}, \mathbf{x}_{33}) = \mathcal{I}_{\mathcal{LK}}(\{0.65, 0.80\}, \{0.90, 1.00\}) = \mathbf{1}_{\mathbb{H}}$; and $A(\mathbf{z_{13}}(\mathbf{x}_3)) = 1.0.$

(d) Since $A(\{0.30, 0.35\}) = 0.3305 \preceq 0.456505 = A(\{0.45, 0.50, 0.65\})$, then we have the next results: $\mathbf{x}_{14} = \{0.30, 0.35\} \preceq^f_A \{0.45, 0.50, 0.65\} = \mathbf{x}_{34}$. If $n = \min(3, 2) = 2$, $\mathbf{z_{13}}(\mathbf{x}_4) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{14}, \mathbf{x}_{34}) = \mathcal{I}_{\mathcal{LK}}(\{0.30, 0.35\}, \{0.45, 0.50, 0.65\}) = \mathbf{1}_{\mathbb{H}}$; Thus, $A(\mathbf{z_{13}}(\mathbf{x}_4)) = 1.0$. Therefore, $\min{(\mathbf{z_{13}}(\mathbf{x}_k))}_{k \in \mathbb{N}_4} = \mathbf{z_{13}}(\mathbf{x}_1).$

(II) Now, for each THFS given in Table 4 related to the third line and the first column ($k = 3; j = 1$) and for all $i \in \mathbb{N}_4$, analogous results from the following expression

$$\mathbf{z}_{31}(\mathbf{x}_k) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{3k}, \mathbf{x}_{1k}) = \begin{cases} \mathbf{1}_{\mathbb{H}}, \text{ if } \mathbf{x}_{3k} \preceq^f_A \mathbf{x}_{1k} \\ A_{n\downarrow}^{(-1)}(1 - A(\mathbf{x}_{3k}) + A(\mathbf{x}_{1k})), \text{ otherwise}; \end{cases} \tag{45}$$

correspond to the achieved data, as reported in the next expressions:
(a) $\mathbf{z_{31}}(\mathbf{x}_1) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{31}, \mathbf{x}_{11}) = \mathbf{1}_{\mathbb{H}}$ and $A(\mathbf{z_{31}}(\mathbf{x}_1)) = 1.0$;
(b) $\mathbf{z_{31}}(\mathbf{x}_2) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{32}, \mathbf{x}_{12}) = \mathbf{1}_{\mathbb{H}}$ and $A(\mathbf{z_{31}}(\mathbf{x}_2)) = 1.0$;
(c) $\mathbf{z_{31}}(\mathbf{x}_3) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{33}, \mathbf{x}_{13}) = \{0.649, 0.901\}$ and $A(\mathbf{z_{31}}(\mathbf{x}_3)) = 0.694091$;
(d) $\mathbf{z_{31}}(\mathbf{x}_4) = \mathcal{I}_{\mathcal{LK}}(\mathbf{x}_{34}, \mathbf{x}_{14}) = \{0.839, 0.795\}$ and $A(\mathbf{z_{31}}(\mathbf{x}_4)) = 0.873995$.
And, therefore, $\min{(\mathbf{z_{31}}(\mathbf{x}_k))}_{k \in \mathbb{N}_4} = \mathbf{z_{31}}(\mathbf{x}_3)$. Thus, combinations of all other lines in Table 4 resulting on the components given as $\mathbf{z_{ij}}(\mathbf{x_k})$, $\forall i \in \mathbb{N}_4, k, j \in \mathbb{N}_7$, which are presented in Table 5 at line i and column j.

Step 5: Now, THFS obtained in Step 4 are ordered increasingly w.r.t. the related $\langle \mathbb{H}, \preceq^f_A \rangle$-, $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders. See, the following sequence which is obtained by taking the THFS obtained in items I. and II. from the previous step, considering the $\langle \mathbb{H}, \preceq_A^f \rangle$-order:

$$\begin{aligned} &\mathbf{z_{13}}(\mathbf{x_1}) \preceq_A^f \mathbf{z_{13}}(\mathbf{x_2}) \preceq_A^f \mathbf{z_{13}}(\mathbf{x_3}) = \mathbf{z_{13}}(\mathbf{x_4}) = \mathbf{1}_{\mathbb{H}}; \text{ and} \\ &\mathbf{z_{31}}(\mathbf{x_3}) \preceq_A^f \mathbf{z_{31}}(\mathbf{x_4}) \preceq_A^f \mathbf{z_{31}}(\mathbf{x_2}) = \mathbf{z_{31}}(\mathbf{x_1}) = \mathbf{1}_{\mathbb{H}}. \end{aligned} \tag{46}$$

Step 6: Finally, each component related to the 4-components of the resulting THFS are aggregated to be compared. The above data obtained from $\mathcal{I}_{\mathcal{LK}}$ are considered in Examples 5.5.1 and 5.5.3 in order to illustrate the action of the following operators $\langle \mathbb{H}, \preceq_A^f \rangle$-$\mathcal{OWA}_\omega$, including $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-$\mathcal{OWA}_\omega$ and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-$\mathcal{OWA}_\omega$. So, we consider the $\langle \mathbb{H}, \preceq_A^f \rangle$-$\mathcal{OWA}_\omega$ aggregation operator in the next items:

(a) Observe, e.g, the data described in the previous step given by Eq. (46a) and Eq. (46b):

$$\begin{aligned} OWA_\omega(\mathbf{z_{13}}) &= \sum_{k=1}^{4} w_k \cdot A\left(\mathbf{z_{13}}(\mathbf{x_k})_{k \in \mathbb{N}_4}\right) \\ &= 0.1 \cdot 0.344945 + 0.2 \cdot 0.6715 + 0.3 \cdot 1.0 + 0.4 \cdot 1.0 \\ &= 0.8688845 \preceq_A^f 0.944209 \\ &= 0.1 \cdot 1.0 + 0.2 \cdot 1.0 + 0.3 \cdot 0.694091 + 0.4 \cdot 0.873995 \\ &= \sum_{k=1}^{4} w_k \cdot A\left(\mathbf{z_{31}}(\mathbf{x_k})_{k \in \mathbb{N}_4}\right) = OWA_\omega(\mathbf{z_{31}}). \end{aligned}$$

where $\omega = \{0.1, 0.2, 0.3, 0.4\}$ and x_i is sorted from the smallest to the greatest element according to Eqs. (46a) and (46b), respectively. So, the related analysis implies that $\mathbf{A}1$ is a better option than $\mathbf{A}3$, denoted as $\mathbf{A1} \sqsupset_A^f \mathbf{A3}$.

(b) By data extracted from Table 5, the comparisons, for all $k \in \mathbb{N}_4$, can be easily verified: If $i = j$ then $\mathbf{Aj} = \mathbf{Ai}$, for all $i, j \in \mathbb{N}_7$. Otherwise, for all $j \in \mathbb{N}_7$, the following holds:

   i. $\mathbf{A1} \sqsupset_A^f \mathbf{Aj}$, because $\mathbf{z}_{j1}(\mathbf{x}_k) \succ_A^f \mathbf{z}_{1j}(\mathbf{x}_k)$ for $j > 1$;
   ii. $\mathbf{A3} \sqsupset_A^f \mathbf{Aj}$, because $\mathbf{z}_{j3}(\mathbf{x}_k) \succ_A^f \mathbf{z}_{3j}(\mathbf{x}_k)$ for $j > 1$, $j \neq 3$;
   iii. $\mathbf{A6} \sqsupset_A^f \mathbf{Aj}$, because $\mathbf{z}_{j6}(\mathbf{x}_k) \succ_A^f \mathbf{z}_{6j}(\mathbf{x}_k)$ for $j > 1$, $j \neq 6$, $j \neq 3$;
   iv. $\mathbf{A4} \sqsupset_A^f \mathbf{Aj}$ because $\mathbf{z}_{j4}(\mathbf{x}_k) \succ_A^f \mathbf{z}_{4j}(\mathbf{x}_k)$, for $j > 1$, $j \neq 6$, $j \neq 3$, $j \neq 4$;
   v. $\mathbf{A2} =_A^f \mathbf{A5} =_A^f \mathbf{A7}$.

Analogous comparisons can be obtained for $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders.

Concluding, Table 8 presents the final comparison relations among the CIM-Applications which is obtained by taking the representable hesitant approach

Table 6 – Partial Overview Analysis of the CIM-Application considering $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-order.

	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$A7$
A1	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.55}; {0.85}; {0.9}; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.55}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.55}; {0.75}; {0.9}; $\mathbf{1}_{\mathbb{H}}$	{0.7}; {0.75}; {0.8}; $\mathbf{1}_{\mathbb{H}}$	{0.45}; {0.45}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.9};{0.55};{0.85}	{0.4}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.4}; {0.55}; $\mathbf{1}_{\mathbb{H}}$	{0.55};$\mathbf{1}_{\mathbb{H}}$;{0.75};{0.9}	{0.7};$\mathbf{1}_{\mathbb{H}}$;{0.75};{0.8}	$\mathbf{1}_{\mathbb{H}}$;{0.45};{0.45};$\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$	{0.895}	{0.88}	{0.85}	{0.875}	{0.86}	{0.835}
A2	{0.95}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.35}; {0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.95}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.5};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.65}; {0.95}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.55}; {0.9}; {0.95}; $\mathbf{1}_{\mathbb{H}}$
	{0.95} ;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.35}; {0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.95}; {0.5}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.5};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.65};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.95}	{0.95};{0.55};{0.9};$\mathbf{1}_{\mathbb{H}}$
	{0.995}	$\mathbf{1}_{\mathbb{H}}$	{0.895}	{0.94}	{0.95}	{0.955}	{0.92}
A3	{0.75};{0.85};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.3};{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.3};{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.5};{0.75};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.65};$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.2}; {0.75}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.75};{0.85}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.3};{0.7}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.7}; {0.3}; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.5};{0.75}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.5};{0.65}	$\mathbf{1}_{\mathbb{H}}$;{0.75};{0.2};$\mathbf{1}_{\mathbb{H}}$
	{0.945}	{0.87}	$\mathbf{1}_{\mathbb{H}}$	{0.87}	{0.9}	{0.88}	{0.87}
A4	{0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.55}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.85}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.55}; {0.6}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.7}; ; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.9}; ; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.7}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.55}	{0.4}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; {0.85}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.55};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.6}	{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.5}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.9};$\mathbf{1}_{\mathbb{H}}$
	{0.97}	{0.955}	{0.91}	$\mathbf{1}_{\mathbb{H}}$	{0.875}	{0.89}	{0.99}
A5	$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.8};{0.9};{0.95};$\mathbf{1}_{\mathbb{H}}$	{0.7}; {0.85}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.9};$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.45}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.9};{0.8};{0.95}	{0.85}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.4}; {0.8}; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.9}	$\mathbf{1}_{\mathbb{H}}$;{0.45};{0.7};$\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$	{0.945}	{0.94}	{0.9}	$\mathbf{1}_{\mathbb{H}}$	{0.99}	{0.885}
A6	{0.9}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.8};{0.8}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.6}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.3}; {0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.9}; {0.85};$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.35}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;{0.9};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.8};{0.8};$\mathbf{1}_{\mathbb{H}}$	{0.7}; {0.6}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.3}; {0.8}; $\mathbf{1}_{\mathbb{H}}$	{0.85};{0.9};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.35};{0.7};$\mathbf{1}_{\mathbb{H}}$
	{0.99};	{0.94}	{0.9}	{0.89}	{0.965}	$\mathbf{1}_{\mathbb{H}}$	{0.875}
A7	{0.6}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.45}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.75}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.9}; {0.95}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.55}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.6}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.45}	{0.4}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.75}	$\mathbf{1}_{\mathbb{H}}$;{0.95}; $\mathbf{1}_{\mathbb{H}}$; {0.9}	{0.55};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.5}	{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.4}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$
	{0.96}	{0.945}	{0.89}	{0.98}	{0.86}	{0.88}	$\mathbf{1}_{\mathbb{H}}$

Table 7 – Partial Overview Analysis of the CIM-Application considering $\langle\mathbb{H}, \preceq_{Lex2}\rangle$-order.

	$A1$	$A2$	$A3$	$A4$	$A5$	$A6$	$A7$
A1	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.8}; {0.95}; $\mathbf{1}_{\mathbb{H}}$	{0.45}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.5}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.55}; {0.6}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.75}; {0.8}; {0.85}; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.55}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.95};$\mathbf{1}_{\mathbb{H}}$; {0.4};{0.8}	{0.45}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.4}; {0.5}; $\mathbf{1}_{\mathbb{H}}$	{0.55};$\mathbf{1}_{\mathbb{H}}$;{0.6};$\mathbf{1}_{\mathbb{H}}$	{0.75};$\mathbf{1}_{\mathbb{H}}$;{0.8};{0.85}	$\mathbf{1}_{\mathbb{H}}$;{0.55};{0.4};$\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$	{0.885}	{0.885}	{0.84}	{0.875}	{0.89}	{0.85}
A2	$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.6};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.55}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$; {0.4}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.6};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.8};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.55};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$	{0.89}	{0.94}	{0.96}	{0.98}	{0.955}
A3	{0.7};{0.8};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.2};{0.5};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.3};{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.4};{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.55}; {0.6};$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.2}; {0.85}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.8};{0.7}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.2};{0.5}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.7}; {0.3}; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.4};{0.7}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.6};{0.55}	$\mathbf{1}_{\mathbb{H}}$;{0.85};{0.2};$\mathbf{1}_{\mathbb{H}}$
	{0.93}	{0.82}	$\mathbf{1}_{\mathbb{H}}$	{0.87}	{0.88}	{0.875}	{0.89}
A4	{0.55}; {0.95}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.35}; {0.9}; {0.9}; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.85}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.5}; {055}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.7}; ; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.9}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	{0.95} ;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.55}	{0.9};$\mathbf{1}_{\mathbb{H}}$;{0.9};{0.35}	{0.4}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; {0.85}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.5};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.55}	{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.4}	$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$;{0.9}; $\mathbf{1}_{\mathbb{H}}$
	{0.945}	{0.885}	{0.91}	$\mathbf{1}_{\mathbb{H}}$	{0.86}	{0.88}	{0.99}
A5	{0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.8};{0.8};{0.8};$\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.9}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.2}; {0.9}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.85}; {0.9};$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.35};{0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$; {0.8};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.8};{0.8};{0.8}	{0.9}; {0.5}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.2}; {0.9};$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.9};$\mathbf{1}_{\mathbb{H}}$;{0.85}	$\mathbf{1}_{\mathbb{H}}$;{0.35};{0.8};$\mathbf{1}_{\mathbb{H}}$
	{0.98}	{0.88}	{0.93}	{0.9}	$\mathbf{1}_{\mathbb{H}}$	{0.965}	{0.895}
A6	{0.9}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.6};{0.9}; {0.95};$\mathbf{1}_{\mathbb{H}}$	{0.6}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.3}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.8}; {0.8};$\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	{0.45}; {0.6}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$
	$\mathbf{1}_{\mathbb{H}}$;{0.9};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.9};{0.6};{0.95}	{0.7}; {0.6}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.3}; {0.7}; $\mathbf{1}_{\mathbb{H}}$	{0.8};$\mathbf{1}_{\mathbb{H}}$;{0.8};$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;{0.45};{0.6};$\mathbf{1}_{\mathbb{H}}$
	{0.99};	{0.925}	{0.9}	{0.87}	{0.94}	$\mathbf{1}_{\mathbb{H}}$	{0.865}
A7	{0.5}; {0.95}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.3}; {0.9}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.4}; {0.8}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.85}; {0.95}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.5}; {0.5}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	{0.35}; {0.7}; $\mathbf{1}_{\mathbb{H}}$; $\mathbf{1}_{\mathbb{H}}$	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$
	{0.95} ;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.5}	{0.9};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.3}	{0.4}; $\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.8}	$\mathbf{1}_{\mathbb{H}}$;{0.85};$\mathbf{1}_{\mathbb{H}}$;{0.95}	{0.5};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.5}	{0.7};$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;{0.35}	$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$;$\mathbf{1}_{\mathbb{H}}$
	{0.94}	{0.91}	{0.9}	{0.975}	{0.85}	{0.875}	$\mathbf{1}_{\mathbb{H}}$

of the Łukasiewicz fuzzy implication w.r.t to the three admissible orders: $\langle \mathbb{H}, \preceq_A^f \rangle$-, $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders. One can see that $A1$ is the best alternative and $A2$ is the worst option, both w.r.t. the three presented $\langle \mathbb{H}, \preceq \rangle$-orders. Besides, it is observed that the $\langle \mathbb{H}, \preceq_A^f \rangle$-order provides as a result, the 5-classes to the set of alternatives. Moreover, in the case of $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-order, the result is presented as a strict increasing chain.

Table 8 – Comparison of $\langle \mathbb{H}, \preceq \rangle$-orders based on the analysis of the CIM-Application

$\langle \mathbb{H}, \preceq \rangle$-order Application	Comparison Relationship for CIM-App
$\langle \mathbb{H}, \preceq_A^f \rangle$-order	$\mathbf{A1} \sqsupset \mathbf{A3} \sqsupset \mathbf{A6} \sqsupset \mathbf{A4} \sqsupset \mathbf{A7} = \mathbf{A5} = \mathbf{A2}$
$\langle \mathbb{H}, \preceq_{Lex1} \rangle$-order	$\mathbf{A1} \sqsupset \mathbf{A3} \sqsupset \mathbf{A6} = \mathbf{A4} = \mathbf{A7} = \mathbf{A5} = \mathbf{A2}$
$\langle \mathbb{H}, \preceq_{Lex2} \rangle$-order	$\mathbf{A1} \sqsupset \mathbf{A3} \sqsupset \mathbf{A7} \sqsupset \mathbf{A6} \sqsupset \mathbf{A5} \sqsupset \mathbf{A4} \sqsupset \mathbf{A2}$

$*$ The Best CIM-Application Choice (CIM-App)

Combining the Łukasiewicz implication operator w.r.t $\langle \mathbb{H}, \preceq \rangle$-order and the hesitant fuzzy triangle product, denoted by the $\lhd$-operator, the results from (WEN; ZHAO; XU, 2019, example 1) present an algorithm which is implemented as a process of alternatives' sorting analysis, resulting on the following comparison: $A1 \succ A3 \succ A6 \succ A4 = A5 = A7 \succ A2$, meaning that A_4, A_5 and A_7 seem belong to the same class w.r.t. $\preceq$-order. This algorithm is based on a comparison between two THFE with the same cardinality, meaning that the shorter one should be extended until its length is the same as the longer one, by adding minimal elements. The results combine the Łukasiewicz implication and the minimum operations to compare two THFE.

7.3 Chapter summary

In this chapter, we described the proposed strategy to solve a ME-MCDM (Multi Expert-Multi Criteria Decision Making) problem considering the $\langle \mathbb{H}, \preceq \rangle$-orders introduced in previous sections. We presented detailed calculation descriptions regarding the $\langle \mathbb{H}, \preceq_A^f \rangle$-order. The other ones should be analogously done. We also extended the proposal methodology to solve this ME-MCDM problem from HFS to THFS, in order to solve the ME-MCDM problem under several alternatives in the selection of a CIM (Computer-Integrated Manufacturing) software, enabling the comparison. Concluding, we presented the final comparison relations among the CIM-Applications which is obtained by taking the representable hesitant approach of the Łukasiewicz fuzzy implication w.r.t to the three admissible orders: $\langle \mathbb{H}, \preceq_A^f \rangle$-, $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders.

8 CONSENSUS MEASURES ON TYPICAL HESITANT FUZZY SETS

This chapter extends the notion of consensus measures on Typical Hesitant Fuzzy Sets. In particular, the properties presented in Definition 3.2.2 are extended to a bounded poset $\mathbf{P}$.

In this proposal, based on formal definition of a consensus measure on the bounded poset $\mathbb{H}$, we formalize $\mathcal{CC}_{\mathcal{A},\mathcal{I}}-$Models to obtain new methodologies of consensus preserving main properties in the context of Typical Hesitant Fuzzy Sets. This study also considers the corresponding extensions of aggregations, implications and fuzzy negations.

8.1 Constructing typical hesitant preference relations

Many applications have used hesitant fuzzy sets in order to elicit preferences in situations where the knowledge is not enough. So, it seems more reasonable to express this hesitancy through a set of values instead of a single one (LIAO; XU; ZENG, 2014, 2015).

In (XIA; XU, 2013) hesitant fuzzy preference relations (HFPR) were formally defined. Thus, for a set $U = \{u_1, \ldots, u_k\}$ of k-alternatives, a Typical Hesitant Fuzzy Preference Relation (THFPR) is defined in the following.

Definition 8.1.1 *A THFPR $\mathcal{R}$ on U, named as $\langle \mathbb{H}, \preceq \rangle$-preference relation, is represented by a THFS on $U \times U$, i.e. by a membership function $\mu_{\mathcal{R}} : U \times U \to \mathbb{H}$. Then, the relation $\mathcal{R} = (X_{ij})_{k\times k}$ whose elements are given by $X_{ij} = \mu_{\mathcal{R}}(u_i, u_j) \in \mathbb{H}$, such that $X_{ij} = \{0.5\}$, for $1 \leq i, j \leq k$, indicating all the possible preference degree(s) of the alternative X_i over X_j. Moreover, an additive $\langle \mathbb{H}, \preceq \rangle$-preference relation is a THFPR $\mathcal{R} = (X_{ij})_{k\times k}$ satisfying the additional condition:*

$$X_{ij} = \mathcal{N}(X_{ji}), \text{ for } 1 \leq i, j \leq k, \tag{47}$$

where $\mathcal{N}$ is an $\langle \mathbb{H}, \preceq \rangle$-negation with $\{0.5\}$ as the fix point.

If $X_{ij} = \{0.5\}$, then it indicates indifference between u_i and u_j and $\{0.5\} \preceq X_{ij}$ indicates that u_i is preferred to u_j.

In the following, additive $\langle \mathbb{H}, \preceq \rangle$-preference relations w.r.t. $\langle \mathbb{H}, \preceq \rangle$-negation are obtained based on aggregation and implication operators.

Proposition 8.1.1 *Let $\mathcal{R}$ be an $\langle \mathbb{H}, \preceq \rangle$-preference relation and $\mathcal{N}$ be an $\langle \mathbb{H}, \preceq \rangle$-negation which has $\{0.5\}$ as a fix point. An additive $\langle \mathbb{H}, \preceq \rangle$-preference relation $\overline{\mathcal{R}} = (\overline{X}_{ij})_{k \times k}$ is defined, for each $1 \leq i, j \leq k$, as follows*

$$\overline{X}_{ij} = \begin{cases} X_{ij}, & \textit{if } X_{ji} \preceq X_{ij}; \\ \mathcal{N}(X_{ji}), & \textit{otherwise.} \end{cases} \tag{48}$$

Proof: If $X_{ji} \preceq X_{ij}$ then $\overline{X}_{ij} = X_{ij}$ and therefore $\overline{X}_{ji} = \mathcal{N}(X_{ij}) = \mathcal{N}(\overline{X}_{ij})$. Otherwise, if $X_{ij} \prec X_{ji}$ then $\overline{X}_{ji} = X_{ji}$ and therefore $\overline{X}_{ij} = \mathcal{N}(X_{ji}) = \mathcal{N}(\overline{X}_{ji})$. □

Definition 8.1.2 *Let $\mathcal{R} = (X_{ij})_{k \times k}$ be THFPR, $\mathcal{A}$ be an extended $\langle \mathbb{H}, \preceq \rangle$-aggregation function, $\mathcal{I}$ be an $\langle \mathbb{H}, \preceq \rangle$-implication and $\mathcal{N}$ be an $\langle \mathbb{H}, \preceq \rangle$-negation. For $1 \leq i, j \leq k$, the operator $\mathcal{R}^* = (X^*_{ij})_{n \times n}$ such that*

$$X^*_{ij} = \begin{cases} \mathcal{A}(\mathcal{I}(X_{i1}, X_{1j}), \ldots, \mathcal{I}(X_{ik}, X_{kj})), & \textit{if } i < j; \\ \mathcal{N}(\mathcal{A}(\mathcal{I}(X_{i1}, X_{1j}), \ldots, \mathcal{I}(X_{ik}, X_{kj}))), & \textit{if } i > j; \\ \{0.5\}, & \textit{if } i = j. \end{cases} \tag{49}$$

defines the $\langle \mathbb{H}, \preceq \rangle$-preference relation w.r.t. $\mathcal{N}$ obtained by $\mathcal{A}$, $\mathcal{I}$, and $\mathcal{R}$.

Proposition 8.1.2 *Let $\mathcal{R} = (X_{ij})_{k \times k}$ be additive $\langle \mathbb{H}, \preceq \rangle$-preference relation, $\mathcal{A}$ be an extended $\langle \mathbb{H}, \preceq \rangle$-aggregation function, $\mathcal{I}$ be an $\langle \mathbb{H}, \preceq \rangle$-implication verifying $\mathcal{I}10$ w.r.t. a $\langle \mathbb{H}, \preceq \rangle$-negation $\mathcal{N}$ which has $\{0.5\}$ as a fix point. The $\langle \mathbb{H}, \preceq \rangle$-preference relation $\mathcal{R}^* = (X^*_{ij})_{n \times n}$ w.r.t. $\mathcal{N}$ obtained by $\mathcal{A}$, $\mathcal{I}$, and $\mathcal{R}$ in Eq.(49) is an additive $\langle \mathbb{H}, \preceq \rangle$-preference relation.*

Proof: Firstly, for $1 \leq i < j \leq 1$, we have that $X^*_{ij} = \mathcal{A}(\mathcal{I}(X_{i1}, X_{1j}), \ldots, \mathcal{I}(X_{ik}, X_{kj}))$. And, in addition, we have that:

$$\begin{aligned} X^*_{ji} &= \mathcal{N}(\mathcal{A}(\mathcal{I}(X_{j1}, X_{1i}), \ldots, \mathcal{I}(X_{jk}, X_{ki}))) \\ &= \mathcal{N}(\mathcal{A}(\mathcal{I}(\mathcal{N}(X_{1i}), \mathcal{N}(X_{j1})), \ldots, \mathcal{I}(\mathcal{N}(X_{ki}), \mathcal{N}(X_{jk})))), \text{ by } \mathcal{I}10 \\ &= \mathcal{N}(\mathcal{A}(\mathcal{I}(X_{i1}, X_{1j}), \ldots, \mathcal{I}(X_{ik}, X_{kj}))) = \mathcal{N}(X^*_{ij}), \text{ by Eq.(47).} \end{aligned}$$

Finally, if $X^*_{ii} = \{0.5\}$ then $X^*_{ii} = \mathcal{N}(X^*_{ii})$. Therefore, Proposition 8.1.2 is verified. □

8.1.1 Consistency on $\langle \mathbb{H}, \preceq \rangle$-preference relations

In the 80's, (TANINO, 1984) introduced an additive diffuse transitivity property, also called complete consistency. Since then, consistency of FPR is associated with the transitivity property (HERRERA-VIEDMA et al., 2004), representing the idea that the preference degree by directly comparing two objectives must be equal to or greater than the degree of preference between these two objectives, using an indirect chain of objectives/actions. This property is desirable to avoid contradictions reflecting in FPR, since the lack of consistency in decision making often leads to inconsistent conclusions. According to (RODRÍGUEZ et al., 2018), the weak transitivity was introduced as the minimum requirement condition which a FPR should satisfy in order to obtain acceptable solutions.

The definition of weak transitivity for HFPR, proposed by (ZHU; XU, 2013), requires that all the possible FPR $R = (x_{ij})_{k\times k}$ belong to a HFPR $\mathcal{R} = (X_{ij})_{k\times k}$, i.e. $x_{ij} \in X_{ij}$ for each $1 \leq i, j \leq k$, which satisfy the condition of weak transitivity yielding the classes of weak/ordinary consistency. In (RODRÍGUEZ et al., 2018), a soft rule to achieve the weak transitivity and ordinary consistency in a HFPR is introduced, since the information provided by experts is hesitant and hence, a degree of contradictory information should be reasonable.

From these points of view, this work extends the weak transitivity and ordinary consistency related to HFPR considering admissible linear orders in $\langle \mathbb{H}, \preceq \rangle$-preference relations.

Definition 8.1.3 *Let $\mathcal{R} = (X_{ij})_{k\times k}$ be an $\langle \mathbb{H}, \preceq \rangle$-preference relation whose elements $X_{ij} \in \mathbb{H}$, such that $X_{ii} = \{0.5\}$ and, for all $i, j, l \in \mathbb{N}_k$, $i \neq j \neq l$, the following condition is verified:*

(WC1) *if $X_{ij} \succeq_{\mathbb{H}} \{0.5\}$ and $X_{jl} \succeq_{\mathbb{H}} \{0.5\}$ then $X_{il} \succeq_{\mathbb{H}} \{0.5\}$;*

thus, $\mathcal{R}$ is called a weak transitivity $\langle \mathbb{H}, \preceq \rangle$-preference relation.

This transitivity can be interpreted as a hesitant information related to three alternatives as follows: If the alternative X_i is preferred to X_l, and X_l is preferred to X_j, then X_i should be preferred or at least equal to X_j.

Main classes of consistency are extended to $\langle \mathbb{H}, \preceq \rangle$-preference relations.

Definition 8.1.4 *An additive $\langle \mathbb{H}, \preceq \rangle$-preference relation $\mathcal{R} = (X_{ij})_{k\times k}$ whose elements $X_{ij} \in \mathbb{H}$, such that $X_{ii} = \{0.5\}$ and, for all $i, j, l \in \mathbb{N}_k$, $i \neq j \neq l$, $\mathcal{R}$ verifies one of the following conditions:*

(OC1) *If $X_{ij} \succ_{\mathbb{H}} \{0.5\}$ and $X_{jl} \succeq_{\mathbb{H}} \{0.5\}$, then $X_{il} \succeq_{\mathbb{H}} \{0.5\}$; or*

(OC2) *If $X_{ij} \succeq_{\mathbb{H}} \{0.5\}$ and $X_{jl} \succ_{\mathbb{H}} \{0.5\}$, then $X_{il} \succeq_{\mathbb{H}} \{0.5\}$; or*

(OC3) *If $X_{ij} = \{0.5\}$ and $X_{jl} = \{0.5\}$, then $X_{il} = \{0.5\}$;*

thus, $\mathcal{R}$ is an ordinal consistency $\langle \mathbb{H}, \preceq \rangle$-preference relation.

Let $\mathcal{R}$ be an $\langle \mathbb{H}, \preceq \rangle$-preference relation and also consider the weak transitivity and the ordinal consistency as proposed in Definitions 8.1.3 and 8.1.4, respectively. If an additive $\langle \mathbb{H}, \preceq \rangle$-preference relation satisfies the weak transitivity and all their THFE (except the diagonal ones) are not equal to the fix point of the related $\langle \mathbb{H}, \preceq \rangle$-negation, then $\mathcal{R}$ also satisfies the ordinal consistency.

Definition 8.1.5 *An additive $\langle \mathbb{H}, \preceq \rangle$-preference relation $\mathcal{R} = (X_{ij})_{k \times k}$ whose elements $X_{ij} \in \mathbb{H}$, such that $X_{ii} = \{0.5\}$ and, for all $i, j, l \in \mathbb{N}_k$, $i \neq j \neq l$, $\mathcal{R}$ verifies one of the following conditions:*

(a) *if $X_{il} \succeq_{\mathbb{H}} \min\{X_{ij}, X_{jl}\}$, then $\mathcal{R}$ satisfies the max-min transitivity;*

(b) *if $X_{il} \succeq_{\mathbb{H}} \max\{X_{ij}, X_{jl}\}$, then $\mathcal{R}$ satisfies the max-max transitivity;*

(c) *if $X_{ij} \succeq_{\mathbb{H}} \{0.5\}$ and $X_{jl} \succeq_{\mathbb{H}} \{0.5\}$, then $X_{il} \succeq_{\mathbb{H}} \min\{X_{ij}, X_{jl}\}$, and $\mathcal{R}$ satisfies the restricted max-min transitivity;*

(d) *if $X_{ij} \succeq_{\mathbb{H}} \{0.5\}$ and $X_{jl} \succeq_{\mathbb{H}} \{0.5\}$, then $X_{il} \succeq_{\mathbb{H}} \max\{X_{ij}, X_{jl}\}$, and $\mathcal{R}$ satisfies the restricted max-max transitivity.*

Observe that max-max transitivity implies max-min transitivity and restricted max-max transitivity implies restricted max-min transitivity.

The weak transitivity is the usual transitivity property that a person should use if one does not want to express inconsistent opinions; however, it is the minimal requirement to find out whether a fuzzy preference relation is consistent or not. On the other hand, the max-max transitivity cannot be verified between X_{ij} and their reciprocal X_{ji}. Also, neither the restricted max-min transitivity nor the restricted max-max transitivity implies reciprocity. Both the additive transitivity and the multiplicative transitivity imply reciprocity. Further studies can be found in (ZHU; XU, 2013; CHICLANA et al., 2008; ALONSO et al., 2008; ZHANG; DONG; XU, 2014; XIA; CHEN, 2015).

Example 8.1.1 *Consider the HFPR presented in (RODRÍGUEZ et al., 2018) and reported below.*

$$\mathcal{R} = \begin{pmatrix} \{0.5\} & \{0.4, 0.6\} & \{0.6\} & \{0.4, 0.6\} \\ \{0.4, 0.6\} & \{0.5\} & \{0.8\} & \{0.4\} \\ \{0.4\} & \{0.2\} & \{0.5\} & \{0.2, 0.3\} \\ \{0.4, 0.6\} & \{0.6\} & \{0.7, 0.8\} & \{0.5\} \end{pmatrix} \tag{50}$$

(i) *Firstly, we consider the consistency analysis based on $\langle \mathbb{H}, \preceq^f_A \rangle$-order, illustrating Definitions 8.1.3 and 8.1.4. One can easily observe the following:*

- *$\mathcal{R}$ is an $\langle \mathbb{H}, \preceq^f_A \rangle$-preference relation which satisfies the weak transitivity and ordinal consistency properties;*
- *Since $X_{32} \preceq^f_A X_{31} \preceq^f_A X_{12}$, meaning that $X_{32} \not\succeq^f_A \min^f_A\{X_{31}, X_{12}\}$ and $X_{32} \not\succeq^f_A \max^f_A\{X_{31}, X_{12}\}$, then $\mathcal{R}$ does not satisfy (a) and (b) properties in Definition 8.1.5;*
- *$\mathcal{R}$ satisfies (c) in Definition 8.1.5;*
- *Since $X_{42} \preceq^f_A X_{43} \preceq^f_A X_{23}$, we have that $X_{43} \not\succeq^f_A \max^f_A\{X_{42}, X_{23}\}$. Then $\mathcal{R}$ does not satisfy (d) in Definition 8.1.5.*

(ii) *The consistency analysis based on $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-order related to the preference relation $\mathcal{R}$ achieved the same results w.r.t. $\langle \mathbb{H}, \preceq^f_A \rangle$-order.*

(iii) *When $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-order is considered, then $\mathcal{R}$ does not satisfy the weak-transitivity, since $X_{24} \not\succeq_{Lex2} \{0.5\}$ despite that equations $X_{21} \succeq_{Lex2} \{0.5\}$ and $X_{14} \succeq_{lex2} \{0.5\}$ are both verified.*

(iv) *According to conditions stated by Definitions 8.1.3 and 8.1.4, there may be a fuzzy preference relation (FPR) which does not satisfy the weak consistency or the ordinal consistency. As well pointed out in (RODRÍGUEZ et al., 2018), in this example, two of the eight FPR do not satisfy the weak transitivity:*

$$\begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.2 \\ 0.4 & 0.6 & 0.8 & 0.5 \end{pmatrix} \quad \begin{pmatrix} 0.5 & 0.4 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.8 & 0.4 \\ 0.4 & 0.2 & 0.5 & 0.3 \\ 0.4 & 0.6 & 0.7 & 0.5 \end{pmatrix}$$

8.2 Consensus measures from $\langle \mathbb{H}, \preceq \rangle$

Based on the formal definition of a consensus measure on $\langle \mathbb{H}, \preceq \rangle$, this section formalizes the consensus measures obtained from $\langle \mathbb{H}, \preceq \rangle$-implications functions and $\langle \mathbb{H}, \preceq \rangle$-aggregation functions denoted by $\mathcal{CC}_{\mathcal{AI}}$-Model. New methods to obtain consensus analysis can be constructed considering the $\mathcal{CC}_{\mathcal{AI}}$-Model, preserving main consensus measures properties in the context of Typical Hesitant Fuzzy Sets. This study also considers the corresponding extensions of aggregations, implications and fuzzy negations.

The generalized notion of consensus measures from $([0,1],\leq)$ to a bounded poset $\mathbb{H} = \langle \mathbb{H}, \preceq \rangle$ is firstly considered, starting with the following definition which straightforwardly extends the usual definition of consensus measure in $[0,1]$:

Definition 8.2.1 *Let $\mathbb{H} = \langle \mathbb{H}, \preceq \rangle$ be a bounded poset. A function $\mathcal{CC} : \bigcup_{n=2}^{\infty} \mathbb{H}^n \to \mathbb{H}$ is a $\mathbb{H}$-valued consensus measure on $\mathbb{H}$ if*

$(\mathcal{C}_{\mathbb{H}}1)$ $\mathcal{CC}(X,\ldots,X) = \mathbf{1}_{\mathbb{H}}$*, for all $X \in \mathbb{H}$ (unanimity);*

$(\mathcal{C}_{\mathbb{H}}2)$ $\mathcal{CC}(\mathbf{0}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}}) = \mathcal{CC}(\mathbf{1}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$ *(minimum consensus for $n = 2$).*

Other properties can be required for $\mathbb{H} = \langle \mathbb{H}, \preceq \rangle$-valued consensus measures on $\mathbb{H}$. Here we just adapt the ones considered in (BOSCH, 2006) in the context of consensus measures on $[0,1]$ (fuzzy consensus measures).

$(\mathcal{C}_{\mathbb{H}}3)$ $\mathcal{CC}(X_1,\ldots,X_n) = \mathcal{CC}(X_{(1)},\ldots,X_{(n)})$ where $(\,)\colon \mathbb{N}_n \to \mathbb{N}_n$ is a permutation (symmetry);

$(\mathcal{C}_{\mathbb{H}}4)$ $\mathcal{CC}(X_1,\ldots,X_n) = \mathbf{0}_{\mathbb{H}}$ when $X_i \in \{\mathbf{0}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}}\}$ for each $i = 1,\ldots,n$ and $\#\{i \in \{1,\ldots,n\} : X_i = \mathbf{0}_{\mathbb{H}}\} = \#\{i \in \mathbb{N}_n\} : X_i = \mathbf{1}_{\mathbb{H}}\}$ (maximum dissension);

$(\mathcal{C}_{\mathbb{H}}5)$ $\mathcal{CC}(X_1,\ldots,X_n) = \mathcal{CC}(X_1,\ldots,X_n,\ldots,X_1,\ldots,X_n)$ (invariance of replication);

$(\mathcal{C}_{\mathbb{H}}6)$ $\mathcal{CC}(X_1,\ldots,X_n) = \mathcal{CC}(\mathcal{N}(X_1),\ldots,\mathcal{N}(X_n))$ (reciprocity for $\mathcal{N}$).

Remark 8.2.1 *Observe that properties $(\mathcal{C}_{\mathbb{H}}2)$, $(\mathcal{C}_{\mathbb{H}}3)$ and $(\mathcal{C}_{\mathbb{H}}5)$ imply property $(\mathcal{C}_{\mathbb{H}}4)$.*

In the next proposition, a method to guarantee the symmetric property for a consensus measure is presented.

Proposition 8.2.1 *Let $(\,)$ be a permutation, $\preceq$ an admissible order on $\mathbb{H}$ and $\mathcal{CC}$ be an $\mathbb{H}$-valued consensus measure on $\mathbb{H}$. Then the function $\mathcal{CC}_S\colon \bigcup_{n=2}^{\infty} \mathbb{H}^n \to \mathbb{H}$ given by*

$$\mathcal{CC}_S(X_1,\ldots,X_n) \;= \mathcal{CC}(X_{(1)},\ldots,X_{(n)}), \tag{51}$$

where $X_{(i)}$ is the i^{th} least element considering $\preceq$-order in the multiset $\{X_1,...,X_n\}$, is an $\langle \mathbb{H}, \preceq \rangle$-valued consensus measure on $\mathbb{H}$ verifying $(\mathcal{C}_{\mathbb{H}}3)$.

Proof: Straightforward. □

8.3 Constructing consensus measures on $\langle \mathbb{H}, \preceq \rangle$-implications

This section presents a methodology to obtain a consensus measure arising from an $\langle \mathbb{H}, \preceq \rangle$-implication, by exploring the additional properties from $(\mathcal{C}_{\mathbb{H}}3)$ to $(\mathcal{C}_{\mathbb{H}}6)$.

8.3.1 $\mathcal{CC}_{\mathcal{A},\mathcal{I}}$-Model

This first method is based on a typical hesitant extended aggregation function $\mathcal{A}$ and an $\langle \mathbb{H}, \preceq \rangle$-implication $\mathcal{I}$.

Theorem 8.3.1 *Let $\mathcal{A}$ be an extended $\langle \mathbb{H}, \preceq \rangle$-aggregation function satisfying $(\mathcal{A}4a)$ and $\mathcal{I}$ be an $\langle \mathbb{H}, \preceq \rangle$-implication verifying $(\mathcal{I}8)$. Then the operator $\mathcal{CC}_{\mathcal{AI}}\colon \bigcup_{n=2}^{\infty} \mathbb{H}^n \to \mathbb{H}$ given by*

$$\mathcal{CC}_{\mathcal{AI}}(X_1, \ldots, X_n) \;=\; \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(X_i, X_j)), \tag{52}$$

is an $\langle \mathbb{H}, \preceq, \mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}} \rangle$-valued consensus measure on $\mathbb{H}$, verifying the conditions:

1. *If $\mathcal{A}$ satisfies $(\mathcal{A}3)$ then $\mathcal{CC}_{\mathcal{AI}}$ satisfies $(\mathcal{C}_{\mathbb{H}}3)$.*
2. *If $\mathcal{A}$ satisfies $(\mathcal{A}3)$ and $\mathcal{I}$ satisfies $(\mathcal{I}10)$ w.r.t. $\mathcal{N}$, then $\mathcal{CC}_{\mathcal{AI}}$ satisfies $(\mathcal{C}_{\mathbb{H}}6)$.*
3. *If $\mathcal{A}$ satisfies ($\mathcal{A}6$) then $\mathcal{CC}_{\mathcal{AI}}$ satisfies $(\mathcal{C}_{\mathbb{H}}5)$.*

Proof: Since $\mathcal{I}$ verifies $(\mathcal{I}8)$ and $\mathcal{A}$ verifies $\mathcal{A}4a$ it holds that:

$$(\mathcal{C}_{\mathbb{H}}1): \mathcal{CC}_{\mathcal{AI}}(X, \ldots, X) = \mathcal{A}(\mathcal{I}(X,X), \ldots, \mathcal{I}(X,X)) = \mathcal{A}(\mathbf{1}_{\mathbb{H}}, \ldots, \mathbf{1}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}};$$
$$(\mathcal{C}_{\mathbb{H}}2): \mathcal{CC}_{\mathcal{AI}}(\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}) = \mathcal{A}(\mathcal{I}(\mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}), \mathcal{I}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}})) = \mathcal{A}(\mathbf{1}_{\mathbb{H}}, \mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}, \text{by } (\mathcal{A}4a).$$

Then, $\mathcal{CC}_{\mathcal{AI}}$ is a $\langle \mathbb{H}, \preceq, \mathbf{0}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}} \rangle$-valued consensus measure on $\mathbb{H}$. And, we also have that the following holds:

$(\mathcal{C}_{\mathbb{H}}3)$: When $\mathcal{A}$ is symmetric the following holds:

$$\begin{aligned}\mathcal{CC}_{\mathcal{AI}}(X_1, \ldots, X_n) &= \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(X_i, X_j)) = \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(X_{(i)}, X_{(j)}))\\ &= \mathcal{CC}_{\mathcal{AI}}(X_{(1)}, \ldots, X_{(n)}).\end{aligned}$$

$(\mathcal{C}_{\mathbb{H}}6)$: If $\mathcal{A}$ satisfies $\mathcal{A}4a$ and $\mathcal{I}$ satisfies $(\mathcal{I}10)$ w.r.t. $\mathcal{N}$, then we obtain the following:

$$\begin{aligned}\mathcal{CC}_{\mathcal{AI}}(X_1, \ldots, X_n) &= \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(X_i, X_j)) = \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(\mathcal{N}(X_j), \mathcal{N}(X_i)))\\ &= \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(\mathcal{N}(X_i), \mathcal{N}(X_j))) = \mathcal{CC}_{\mathcal{AI}}(\mathcal{N}(X_1), \ldots, \mathcal{N}(X_n)).\end{aligned}$$

$(\mathcal{C}_{\mathbb{H}}5)$: And finally, if $\mathcal{A}$ satisfies ($\mathcal{A}6$) the following holds:

$$\begin{aligned}\mathcal{CC}_{\mathcal{AI}}(X_1, \ldots, X_n) &= \mathcal{A}^n_{i,j=1, i\neq j}(\mathcal{I}(X_i, X_j)) \text{ by Eq.(52)}\\ &= \mathcal{A}^n_{i,j=1, i\neq j}\left(\underbrace{\mathcal{I}(X_i, X_j), \ldots, \mathcal{I}(X_i, X_j)}_{n-times}\right) \text{ by } (\mathcal{A}6)\\ &= \mathcal{CC}_{\mathcal{AI}}(\underbrace{X_1, \ldots, X_n, \ldots, X_1, \ldots, X_n}_{n-times}) \text{ by Eq.(52).}\end{aligned}$$

Concluding, Theorem 8.3.1 is verified. □

Remark 8.3.1 *One can observe that, when $\mathcal{A}$ satisfies ($\mathcal{A}3$), ($\mathcal{A}4a$) and ($\mathcal{A}6$) then $\mathcal{CC}_{\mathcal{AI}}$ satisfies ($\mathcal{C}_{\mathbb{H}}4$).*

Corollary 8.3.1 *Let $\mathcal{T}_{LK}$ and $\mathcal{I}_{LK}$ be the Łukasiewicz $\langle\mathbb{H},\preceq_{Lex1}\rangle$-t-norm and the Łukasiewicz $\langle\mathbb{H},\preceq_{Lex1}\rangle$-implication. Then $\mathcal{CC}_{\mathcal{T}_{LK},\mathcal{I}_{LK}}\colon \bigcup_{n=2}^{\infty}\mathbb{H}^n \to \mathbb{H}$ given by*

$$\mathcal{CC}_{\mathcal{T}_{LK},\mathcal{I}_{LK}}(X_1,\ldots,X_n) = \mathop{\mathcal{T}_{LK}}_{i,j=1,i\neq j}^{n}(\mathcal{I}_{LK}(X_i,X_j)), \tag{53}$$

is an $\langle\mathbb{H},\preceq_{Lex1},\mathbf{0}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}}\rangle$-valued consensus measure on $\mathbb{H}$. In addition, $\mathcal{CC}_{\mathcal{T}_{LK},\mathcal{I}_{LK}}$ verifies the conditions:

1. *($\mathcal{C}_{\mathbb{H}}3$) since $\mathcal{T}_{LK}$ is symmetric;*
2. *($\mathcal{C}_{\mathbb{H}}6$) since $\mathcal{T}_{LK}$ is symmetric and $\mathcal{I}_{LK}$ satisfies ($\mathcal{I}10$); and*
3. *($\mathcal{C}_{\mathbb{H}}5$) since $\mathcal{T}_{LK}$ satisfies ($\mathcal{A}4a$);*

meaning that it also verifies ($\mathcal{C}_{\mathbb{H}}4$).

8.3.2 $\mathcal{CC}_{\min,\mathcal{I}}$-Model

This method considers the minimum typical hesitant aggregation ($\mathcal{T}_{\min} = \min$) as presented in Example 5.4.1, and an $\langle\mathbb{H},\preceq\rangle$-implication $\mathcal{I}$.

Theorem 8.3.2 *Let $\mathcal{A}$ be an extended $\langle\mathbb{H},\preceq\rangle$-aggregation function satisfying ($\mathcal{A}4a$) and $\mathcal{I}$ be an $\langle\mathbb{H},\preceq\rangle$-implication verifying ($\mathcal{I}8$). The operator $\mathcal{CC}_{\min,I}\colon \bigcup_{n=2}^{\infty}\mathbb{H}^n \to \mathbb{H}$ given by*

$$\mathcal{CC}_{\min,I}(X_1,\ldots,X_n) = \begin{cases} \mathcal{I}(X_1,X_2)\wedge\mathcal{I}(X_2,X_1), & \text{if } n=2\\ \mathcal{CC}_{\min,I}(X_1,\mathcal{CC}_{\min,I}(X_2,\ldots,X_n)), & \text{if } n>2, \end{cases} \tag{54}$$

is an $\langle\mathbb{H},\preceq\rangle$-valued consensus measure on $\mathbb{H}$ verifying the condition:

If $\mathcal{I}$ satisfies ($\mathcal{I}10$) w.r.t. $\mathcal{N}$ then $\mathcal{CC}_{\min,I}$ satisfies ($\mathcal{C}_{\mathbb{H}}6$).

Proof: Let $\langle\mathbb{H},\preceq\rangle$ be a bounded lattice related to an admissible $\preceq$- order on $\mathbb{H}$. Let $\mathcal{I}$ be a $\langle\mathbb{H},\preceq\rangle$-implication verifying ($\mathcal{I}8$).
($\mathcal{CC}1$): Clearly, by property ($\mathcal{I}8$), $\mathcal{CC}_{\min,I}(X,X) = \mathcal{I}(X,X) = \mathbf{1}_{\mathbb{H}}$, and, the inductive hypothesis is that for $k \geq 2$, $\mathcal{CC}_{\min,I}(\underbrace{X,\ldots,X}_{k-times}) = \mathbf{1}_{\mathbb{H}}$. Then, we have that $\mathcal{CC}_{\min,I}(\underbrace{X,\ldots,X}_{(k+1)-times}) = \mathcal{CC}_{\min,I}(X,\mathcal{CC}_{\min,I}(\underbrace{X,\ldots,X}_{k-times})) = \mathcal{I}(X,\mathbf{1}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}}$. So, by induction,

for each $n \geq 2$ we obtain the following: $\mathcal{CC}_{\min,I}(\underbrace{X,\ldots,X}_{n-times}) = \mathbf{1}_{\mathbb{H}}$.

$(\mathcal{CC}2)$: $\mathcal{CC}_{\min,I}(\mathbf{0}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}}) = \mathcal{I}(\mathbf{0}_{\mathbb{H}},\mathbf{1}_{\mathbb{H}}) \wedge \mathcal{I}(\mathbf{1}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) = \mathbf{1}_{\mathbb{H}} \wedge \mathbf{0}_{\mathbb{H}} = \mathbf{0}_{\mathbb{H}}$. Analogously, it is proved that $\mathcal{CC}_{\min,I}(\mathbf{1}_{\mathbb{H}},\mathbf{0}_{\mathbb{H}}) = \mathbf{0}_{\mathbb{H}}$. Then, $\mathcal{CC}_{\min,I}$ is a $\langle \mathbb{H}, \preceq \rangle$-valued consensus measure on $\mathbb{H}$.

$(\mathcal{C}_{\mathbb{H}}6)$: If $\mathcal{I}$ satisfies $(\mathcal{I}10)$ w.r.t. $\mathcal{N}$, then for $n = 2$ we obtain the following results:

$$\begin{aligned}
\mathcal{CC}_{\min,I}(X_1,X_2) &= \mathcal{I}(X_1,X_2) \wedge \mathcal{I}(X_2,X_1) \\
&= \mathcal{I}(\mathcal{N}(X_2),\mathcal{N}(X_1)) \wedge \mathcal{I}(\mathcal{N}(X_1),\mathcal{N}(X_2)) \\
&= \mathcal{I}(\mathcal{N}(X_1),\mathcal{N}(X_2)) \wedge \mathcal{I}(\mathcal{N}(X_2),\mathcal{N}(X_1)) = \mathcal{CC}_{\min,I}(\mathcal{N}(X_1),\mathcal{N}(X_2)).
\end{aligned}$$

Assume now, that the hypothesis holds for n. For $n+1$ we obtain that:

$$\begin{aligned}
&\mathcal{CC}_{\min,I}(X_1,\ldots,X_{n+1}) = \mathcal{CC}_{\min,I}(X_1,\mathcal{CC}_{\min,I}(X_2,\ldots,X_{n+1})) \\
&\quad = \mathcal{I}(X_1,\mathcal{CC}_{\min,I}(X_2,\ldots,X_{n+1})) \wedge \mathcal{I}(\mathcal{CC}_{\min}(X_2,\ldots,X_{n+1}),X_1) \\
&\quad = \mathcal{I}(\mathcal{N}(\mathcal{CC}_{\min,I}(X_2,\ldots,X_{n+1})),\mathcal{N}(X_1)) \wedge \mathcal{I}(\mathcal{N}(X_1),\mathcal{N}(\mathcal{CC}_{\min,I}(X_2,\ldots,X_{n+1}))) \\
&\quad = \mathcal{I}(\mathcal{N}(X_1),\mathcal{N}(\mathcal{CC}_{\min,I}(X_2,\ldots,X_{n+1})) \wedge \mathcal{I}(\mathcal{N}(\mathcal{CC}_{\min,I}(X_2,\ldots,X_{n+1}),\mathcal{N}(X_1))) \\
&\quad = \mathcal{I}(\mathcal{N}(X_1),\mathcal{CC}_{\min,I}(\mathcal{N}(X_2),\ldots,\mathcal{N}(X_{n+1}))) \\
&\qquad\qquad \wedge \mathcal{I}(\mathcal{CC}_{\min,I}(\mathcal{N}(X_2),\ldots,\mathcal{N}(X_{n+1})),\mathcal{N}(X_1)), \text{ by HI} \\
&\quad = \mathcal{CC}_{\min,I}(\mathcal{N}(X_1),\mathcal{CC}_{\min,I}(\mathcal{N}(X_2),\ldots,\mathcal{N}(X_{n+1}))) \\
&\quad = \mathcal{CC}_{\min,I}(\mathcal{N}(X_1),...,\mathcal{N}(X_{n+1}).
\end{aligned}$$

Concluding, Theorem 8.3.1 is verified. □

Corollary 8.3.2 *Let $\mathcal{A}$ be an extended $\langle \mathbb{H}, \preceq \rangle$-aggregation function satisfying $(\mathcal{A}4a)$ and $\mathcal{I}$ be an $\langle \mathbb{H}, \preceq \rangle$-implication verifying $(\mathcal{I}8)$. The function $\mathcal{CC}_I \colon \bigcup_{n=2}^{\infty} \mathbb{H}^n \to \mathbb{H}$ given by*

$$\mathcal{CC}_I(X_1,\ldots,X_n) = \begin{cases} \mathcal{I}(X_{(2)},X_{(1)}) & \text{if } n = 2 \\ \mathcal{CC}_I(X_{(1)},\mathcal{CC}_I(X_{(2)},\ldots,X_{(n)})) & \text{if } n > 2\,, \end{cases} \tag{55}$$

where $X_{(i)}$ is the i^{th} least element w.r.t. $\preceq$-order in the multiset $\{X_1,...,X_n\}$, is an $\langle \mathbb{H}, \preceq \rangle$-valued consensus measure on $\mathbb{H}$ verifying $(\mathcal{C}_{\mathbb{H}}3)$ and :

If $\mathcal{I}$ satisfies $(\mathcal{I}10)$ w.r.t. $\mathcal{N}$ then $\mathcal{CC}_I$ satisfies $(\mathcal{C}_{\mathbb{H}}6)$.

Proof: Straightforward from Proposition 8.2.1 and Theorem 8.3.2. □

Remark 8.3.2 *Observe that $\mathcal{CC}_I(X_1,X_2) = \mathcal{CC}_{\min,I}(X_1,X_2)$.*

8.3.3 Illustrating $\mathcal{CC}_{\mathcal{AI}}$-consensus model on $\langle \mathbb{H}, \preceq \rangle$-orders

Firstly, an algorithmic method is described in this section, followed by an application in ME-MCDM providing ratings for styles of craft beers.

8.3.3.1 $\mathcal{CC}_{\mathcal{AI}}$-consensus model methodology

The $\mathcal{CC}_{\mathcal{AI}}$-consensus model introduces a method to generate the HFPR based on the consistency analysis and the consensus measure introduced in previous sections.

The two main steps consolidating $\mathcal{CC}_{\mathcal{AI}}$-consensus model are listed below:

1. Preconditional steps

   (1.1) Selection of the $\langle \mathbb{H}, \preceq \rangle$-order;

   (1.2) Election of $\mathcal{CC}_{\mathcal{AI}}$-consensus model, determining the corresponding negation, implication and aggregation operators;

   (1.3) Construction of the matrix $\mathcal{R} = (X_{ij})_{n\times n}$ based on the $\langle \mathbb{H}, \preceq \rangle$-preference relation associate to the n-alternatives;

   (1.4) Consistency analysis of relation $\mathcal{R}$, eliciting pairwise comparison between alternatives to eliminate inconsistency which support incomplete fuzzy preference relations and missing information;

   (1.5) Definition of corresponding additive matrix structure $\overline{\mathcal{R}} = (\overline{X}_{ij}^{(k)})_{n\times n}$ related to the $\langle \mathbb{H}, \preceq \rangle$-preference relation, calculating the THFE for all positions in the matrix structure based on Eq. (49);

   (1.6) Determination of the α-level criteria to achieve an α-level consensus in the additive $\langle \mathbb{H}, \preceq_A^f \rangle$-preference relation $\overline{\mathcal{R}}$.

2. Iterative Steps: Construction of $\overline{\mathcal{R}}^{(k)} = (\overline{X}_{ij}^{(k)})_{n\times n}$, for $k = 1$.

   (2.1) Taking $k = 1$;

   (2.2) Calculation of the k-iteration of $\overline{\mathcal{R}}^{(i)}$;

   (2.3) Calculation of the k-iteration of $\mathcal{CC}_{\mathcal{AI}} = (Y_{ij})_{n\times n}$ based on Eq. (52) and operators defined in previous item (1.2);

   (2.4) Compare the α-level between the additive $\langle \mathbb{H}, \preceq \rangle$-preference relations $\overline{\mathcal{CC}}^{(k)}$ and $\overline{\mathcal{CC}}^{(k-1)}$:

      i. If there exist $\overline{X}_{ij}^{(k)} \preceq_{\mathbb{H}} \{\alpha\}$, for $i < j$ then $k = k + 1$, and return to Step (2.2);

      ii. Otherwise, finish the algorithm process.

8.3.3.2 $\mathcal{CC}_{\mathcal{AI}}$-consensus model application in ME-MCDM problem

Considering a group of four friends providing ratings for three styles of craft beers (as shown in Table 3) as well as comparing the average, which was firstly introduced in (MATZENAUER et al., 2019/08). One can also see that while everyone partially agrees that the craft beer Sour style is not very good, and the Pale Ale style is not too bad, there is a lack of consensus regarding the Weiss style.

The corresponding preference matrices R_{F_1}, R_{F_2}, R_{F_3} and R_{F_4} related to all craft beer styles described in Section 3.2, Example 3.2.1.

And, the THFS containing all the preferences, for each ij-position on the Example 3.2.1, four preference matrices result on the matrix structure $\mathcal{R} = (X_{ij})_{4\times 4}$ presented as an additive HFPR as given in Eq.(56) below:

$$\mathcal{R} = \begin{pmatrix} \{0.5\} & \{0.1; 0.3; 0.4\} & \{0.1; 0.2; 0.3; 0.4\} \\ \{0.6; 0.7; 0.9\} & \{0.5\} & \{0.2; 0.3; 0.4; 0.9\} \\ \{0.6; 0.7; 0.8; 0.9\} & \{0.1; 0.6; 0.7; 0.8\} & \{0.5\} \end{pmatrix} \quad (56)$$

In the consistency analysis, $\mathcal{R}$ is an $\langle \mathbb{H}, \preceq^f_A \rangle$- ($\langle \mathbb{H}, \preceq_{Lex1} \rangle$-) preference relation preserving the weak transitivity and ordinal consistency. And it also satisfies (a) and (c) properties in Definition 8.1.5 w.r.t. $\langle \mathbb{H}, \preceq^f_A \rangle$- $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-order. However, it does not verify properties from (b) e (d) in Definition 8.1.5, since $\max\{X_{32}, X_{21}\} \succeq^f_A X_{31}$ ($\max\{X_{32}, X_{21}\} \succeq_{Lex1} X_{31}$).

Finally, we also have that $\mathcal{R}$ is an $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-preference relation preserving the weak transitivity and ordinal consistency and property (a) in Definition 8.1.5. But it does not satisfy (b) since $\max\{X_{13}, X_{32}\} \succeq_{Lex2} X_{12}$. Analogous results follows from (c) and (d) analysis.

8.3.3.3 $\mathcal{CC}_{\mathcal{AI}}$-consensus model application based on $\langle \mathbb{H}, \preceq^f_A \rangle$-orders

From the matrix $\mathcal{R} = (X_{ij})_{3\times 3}$ related to $\langle \mathbb{H}, \preceq^f_A \rangle$-preference relation $\mathcal{R}$, shown in Section 8.3.3.2, the additive matrix structure $\overline{\mathcal{R}} = (\overline{X}_{ij})_{3\times 3}$ is reported as follows:

$$\overline{\mathcal{R}} = \begin{pmatrix} \{0.5\} & \{0.321\} & \{0.3211\} \\ \{0.6; 0.7; 0.9\} & \{0.5\} & \{0.2; 0.3; 0.4; 0.9\} \\ \{0.6; 0.7; 0.8; 0.9\} & \{0.7651\} & \{0.5\} \end{pmatrix} \quad (57)$$

Take α=0.6 as a criteria to achieve an α-level consensus in the $\langle \mathbb{H}, \preceq^f_A \rangle$-preference relation $\overline{\mathcal{R}}$, meaning that $\overline{X}_{ij} \succeq^f_A \{0.6\}$, for all $i > j$.

The k-iteration over $\overline{\mathcal{R}}$ is performed, for $k \in \mathbb{N}_3$, resulting on the $\overline{\mathcal{R}}^{(k)} = (\overline{X}^{(k)}_{ij})_{3\times 3}$

matrices, described in Table 9. These components in $\overline{\mathcal{R}}^{(1)}$ are obtained applying Eq. (52) from Theorem 8.3.1. See below, how the resulting THFS in 12-position in $\overline{\mathcal{R}}^{(1)}$ is calculated:

$$\overline{X}_{12}^{(1)} = \mathcal{T}_{LK}(\mathcal{I}_{LK}(\overline{X}_{11}, \overline{X}_{12}), \mathcal{I}_{LK}(\overline{X}_{12}, \overline{X}_{22}), \mathcal{I}_{LK}(\overline{X}_{13}, \overline{X}_{32})),$$

when the corresponding implications returned the following data:

$$\mathcal{I}_{LK}(\overline{X}_{11}, \overline{X}_{12}) = \mathcal{I}_{LK}(\{0.5\}, \{0.321\}) = \{0.821\};$$
$$\mathcal{I}_{LK}(\overline{X}_{12}, \overline{X}_{22}) = \mathcal{I}_{LK}(\{0.321\}, \{0.5\}) = \mathbf{1}_{\mathbb{H}};$$
$$\mathcal{I}_{LK}(\overline{X}_{13}, \overline{X}_{32}) = \mathcal{I}_{LK}(\{0.3211\}, \{0.761\}) = \mathbf{1}_{\mathbb{H}}.$$

So, $\overline{X}_{12}^{(1)} = \mathcal{T}_{LK}(\{0.821\}, \mathbf{1}_{\mathbb{H}}, \mathbf{1}_{\mathbb{H}}) = \{0.821\}$. And thus, by comparing and applying the Łukasiewicz typical hesitant fuzzy t-norm, the consensus is described by the matrix structure $\mathcal{CC}^{(1)} = (\overline{Y}_{ij}^{(1)})_{3\times 3}$, as components of matrix $\mathcal{CC}^{(1)}$ given in line 1, column 2 of Table 9.

Observing, by Eq. (55), the $\overline{Y}_{12}^{(1)}$ component is calculated as follows:

$$\begin{aligned}\overline{Y}_{12}^{(1)} &= \mathcal{T}_{LK}(\mathcal{I}_{LK}(\overline{X}_{12}, \overline{X}_{12}^{(1)}), \mathcal{I}_{LK}(\overline{X}_{12}^{(1)}, \overline{X}_{12}))\\ &= \mathcal{T}_{LK}(\mathcal{I}_{LK}(\{0.821\}, \{0.321\}), \mathcal{I}_{LK}(\{0.321\}, \{0.821\})) = \{0.5\} \preceq_A^f \{0.6\}.\end{aligned}$$

Analogously, the other $\overline{Y}_{ij}^{(1)}$ components can be obtained. At 1-iteration, the estimate α-level consensus is achieved just to the THFS in the $\overline{Y}_{23}^{(1)}$ component. Thus, it is necessary to generate a new iteration from $\overline{R}^{(1)}$ resulting on $\overline{R}^{(2)}$. So, at 2nd-iteration, the estimate α-level consensus is achieved just to the THFS in the $\overline{Y}_{12}^{(1)}$ and $\overline{Y}_{23}^{(1)}$ components in $\mathcal{CC}^{(2)}$ matrix structure. Thus, it is necessary to generate a new iteration from $\overline{R}^{(2)}$ resulting on $\overline{R}^{(3)}$. Finally, at the third iteration, $\alpha = 0.69$ is achieved as the consensus rate, meaning that all $\overline{Y}_{ij}^{(1)} \succeq_A^f \{0.6\}$, for all $i < j$. This can be interpreted as a consensus rate up to 69%.

8.3.3.4 $\mathcal{CC}_{\mathcal{AI}}$-consensus model application based on $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-order

From the matrix $\mathcal{R} = (X_{ij})_{3\times 3}$ related to $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-preference relation $\mathcal{R}$, shown in Section 8.3.3.2, the additive matrix structure $\overline{\mathcal{R}} = (\overline{X}_{ij})_{3\times 3}$ as defined in sequence:

$$\overline{\mathcal{R}} = \begin{pmatrix} \{0.5\} & \{0.4\} & \{0.4\} \\ \{0.6; 0.7; 0.9\} & \{0.5\} & \{0.2; 0.3; 0.4; 0.9\} \\ \{0.6; 0.7; 0.8; 0.9\} & \{0.8\} & \{0.5\} \end{pmatrix}$$

Then, applying the algorithm described in Section 8.3.3.1 and considering the same

Table 9 – $\mathcal{CC}_{\mathcal{AI}}$-consensus model application based on $\langle \mathbb{H}, \preceq_A^f \rangle$-order.

k	$\overline{\mathcal{R}}^{(k)}$	$\mathcal{CC}^{(k)}$
1	$\begin{pmatrix} \{0.5\} & \{0.821\} & \{0.735\} \\ \{0.19;0.7\} & \{0.5\} & \{0.37;0.7\} \\ \{0.25;0.6\} & \{0.623\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \{0.5\} & \{0.56;0.8\} \\ \{0.5\} & \mathbf{1}_{\mathbb{H}} & \{0.8579\} \\ \{0.56;0.8\} & \{0.8579\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$
2	$\begin{pmatrix} \{0.5\} & \{0.57;0.6\} & \{0.321\} \\ \{0.433\} & \{0.5\} & \{0.37;0.7\} \\ \{0.69;0.7\} & \{0.623\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \{0.746\} & \{0.56;0.8\} \\ \{0.746\} & \mathbf{1}_{\mathbb{H}} & \{\mathbf{1}_{\mathbb{H}}\} \\ \{0.56;0.8\} & \{\mathbf{1}_{\mathbb{H}}\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$
3	$\begin{pmatrix} \{0.5\} & \{0.57;0.6\} & \{0.631\} \\ \{0.433\} & \{0.5\} & \{0.37;0.7\} \\ \{0.39;0.6\} & \{0.623\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} & \{0.69\} \\ \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} & \{\mathbf{1}_{\mathbb{H}}\} \\ \{0.69\} & \{\mathbf{1}_{\mathbb{H}}\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$

α-criteria, the consensus is reached in the 3rd-iteration. See the results in Table 10.

8.3.3.5 $\mathcal{CC}_{\mathcal{AI}}$-consensus model application based on $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-order

Analogously, starting with the matrix $\mathcal{R} = (X_{ij})_{3\times 3}$ related to $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-preference relation $\mathcal{R}$, shown in Section 8.3.3.2, the additive matrix structure $\overline{\mathcal{R}} = (\overline{X}_{ij})_{3\times 3}$ is given in the following:

$$\overline{\mathcal{R}} = \begin{pmatrix} \{0.5\} & \{0.1\} & \{0.1\} \\ \{0.6;0.7;0.9\} & \{0.5\} & \{0.2;0.3;0.4;0.9\} \\ \{0.6;0.7;0.8;0.9\} & \{0.1\} & \{0.5\} \end{pmatrix}$$

And, by application of the algorithm described in Section 8.3.3.1 and the same α-criteria, the consensus is reached in the 3rd-iteration. See the results in Table 11.

Remark 8.3.3 *Based on results from Corollary 8.3.2, the application based on $\mathcal{CC}_{\min,\mathcal{I}}$-Model, as defined in Theorem 8.3.2 achieves the same result consensus.*

8.4 Chapter summary

In this chapter, we extended the notion of consensus measures on Typical Hesitant Fuzzy Sets, based on formal definition of a consensus measure on the

Table 10 – $\mathcal{CC}_{\mathcal{AI}}$-consensus model application based on $\langle \mathbb{H}, \preceq_{Lex1} \rangle$-order.

k	$\overline{\mathcal{R}}^{(k)}$	$\mathcal{CC}^{(k)}$
1	$\begin{pmatrix} \{0.5\} & \{0.9\} & \{0.5\} \\ \{0.1\} & \{0.5\} & \{0.5\} \\ \{0.5\} & \{0.5\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \{0.5\} & \{0.9\} \\ \{0.5\} & \mathbf{1}_{\mathbb{H}} & \{0.7\} \\ \{0.9\} & \{0.7\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$
2	$\begin{pmatrix} \{0.5\} & \{0.6\} & \{0.5\} \\ \{0.4\} & \{0.5\} & \{0.5\} \\ \{0.5\} & \{0.5\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \{0.7\} & \mathbf{1}_{\mathbb{H}} \\ \{0.7\} & \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} \\ \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$
3	$\begin{pmatrix} \{0.5\} & \{0.57; 0.6\} & \{0.631\} \\ \{0.433\} & \{0.5\} & \{0.37; 0.7\} \\ \{0.39; 0.6\} & \{0.623\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} & \{0.69\} \\ \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} & \{\mathbf{1}_{\mathbb{H}}\} \\ \{0.69\} & \{\mathbf{1}_{\mathbb{H}}\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$

bounded poset $\mathbb{H}$. We formalised the $\mathcal{CC}_{\mathcal{A},\mathcal{I}}-$Models to obtain new methodologies of consensus preserving main properties in the context of Typical Hesitant Fuzzy Sets, also considering the corresponding extensions of aggregations, implications and fuzzy negations.

The weak transitivity and ordinary consistency related to HFPR were also extended considering admissible linear orders in $\langle \mathbb{H}, \preceq \rangle$-preference relations.

And, based on the formal definition of a consensus measure on $\langle \mathbb{H}, \preceq \rangle$, the $\mathcal{CC}_{\mathcal{AI}}$-Model was formalised from $\langle \mathbb{H}, \preceq \rangle$-implications functions and $\langle \mathbb{H}, \preceq \rangle$-aggregation functions. New methods to obtain consensus analysis were constructed considering this $\mathcal{CC}_{\mathcal{AI}}$-Model, and preserving main consensus measures properties in the context of Typical Hesitant Fuzzy Sets.

Then, was presented a methodology to obtain a consensus measure arising from an $\langle \mathbb{H}, \preceq \rangle$-implication, where the first method was based on a typical hesitant extended aggregation function $\mathcal{A}$ and an $\langle \mathbb{H}, \preceq \rangle$-implication $\mathcal{I}$; and the second method considers the minimum typical hesitant aggregation ($\mathcal{T}_{\min} = \min$) and an $\langle \mathbb{H}, \preceq \rangle$-implication $\mathcal{I}$.

Some illustrating examples of $\mathcal{CC}_{\mathcal{AI}}$- consensus model on $\langle \mathbb{H}, \preceq \rangle$-orders were discussed. We presented an algorithmic method followed by an application in ME-MCDM providing ratings for styles of craft beers.

Table 11 – $\mathcal{CC}_{\mathcal{AI}}$-consensus model application based on $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-order.

k	$\overline{\mathcal{R}}^{(k)}$	$\mathcal{CC}^{(k)}$
1	$\begin{pmatrix} \{0.5\} & \{0.6\} & \{0.6\} \\ \{0.4\} & \{0.5\} & \{0.2\} \\ \{0.4\} & \{0.8\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \{0.5\} & \{0.5\} \\ \{0.5\} & \mathbf{1}_{\mathbb{H}} & \{0.3\} \\ \{0.5\} & \{0.3\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$
2	$\begin{pmatrix} \{0.5\} & \{0.9\} & \{0.5\} \\ \{0.1\} & \{0.5\} & \{0.7\} \\ \{0.5\} & \{0.3\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \{0.7\} & \{0.9\} \\ \{0.7\} & \mathbf{1}_{\mathbb{H}} & \{0.5\} \\ \{0.9\} & \{0.5\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$
3	$\begin{pmatrix} \{0.5\} & \{0.9\} & \{0.5\} \\ \{0.1\} & \{0.5\} & \{0.8\} \\ \{0.5\} & \{0.2\} & \{0.5\} \end{pmatrix}$	$\begin{pmatrix} \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} \\ \mathbf{1}_{\mathbb{H}} & \mathbf{1}_{\mathbb{H}} & \{0.9\} \\ \mathbf{1}_{\mathbb{H}} & \{0.9\} & \mathbf{1}_{\mathbb{H}} \end{pmatrix}$

9 FINAL CONSIDERATIONS

In this work, new ideas in THFE are investigated and developed under the scope of an arbitrary order, allowing the possibility of comparisons of THFE with different cardinalities. A class of admissible linear $\langle \mathbb{H}, \preceq_A^f \rangle$-orders is also presented, when A is an increasing aggregation function and f satisfies the injective-cardinality property. Two other admissible linear orders are considered: $\langle \mathbb{H}, \preceq_{Lex1} \rangle$- and $\langle \mathbb{H}, \preceq_{Lex2} \rangle$-orders, that allowed us to introduce a formal definition of hesitant fuzzy operators, such as $\langle \mathbb{H}, \preceq \rangle$-aggregation functions and $\langle \mathbb{H}, \preceq \rangle$-negations, including their respective important properties. Emphasizing as main results, we have the generation of $\langle \mathbb{H}, \preceq \rangle$-negations from fuzzy negations, also presenting some interesting examples.

Here, the contextual theoretical research on the properties of $\langle \mathbb{H}, \preceq \rangle$-implications is related to the monotonicity analysis, which is restricted to the first place antitonicity and second place isotonicity, but also including the identity and exchange principles, the left and right boundary conditions, the contrapositive symmetry and ordering properties. Some additional examples are also presented, mainly related to natural negations obtained from $\langle \mathbb{H}, \preceq \rangle$-implication functions.

Based on such classes of $\langle \mathbb{H}, \preceq \rangle$-orders, expressions for the main examples of aggregation functions and fuzzy implications are presented. Furthermore, the representability of such operators is obtained from generation of $\langle \mathbb{H}, \preceq \rangle$-implications as an order-preserving structure of main implication properties. Besides, is introduced a formal definition for the representability of those negations, by constructing a method to obtain $\langle \mathbb{H}, \preceq \rangle$-implications from $\langle \mathbb{H}, \preceq \rangle$-aggregations.

Another relevant contribution illustrating our theoretical results is an algorithmic solution for an ME-MCDC problem, which used $\langle \mathbb{H}, \preceq \rangle$-operators and took into account the selection of a CIM-software. By applying the Łukasiewicz implication, the example reported an ME-MCDM problem in a CIM-application, which could be analysed from three distinct comparisons based on $\langle \mathbb{H}, \preceq \rangle$-operators.

This work also presents improved consensus-based procedures based on admissible $\langle \mathbb{H}, \preceq \rangle$-orders, handling Multi Expert-Multi Criteria Decision Making (ME-MCDM) problems and using consistent $\langle \mathbb{H}, \preceq \rangle$-preference relations (HFPR). At the first level, the consistence analysis considers the weak transitivity and ordinal consistency properties in $\langle \mathbb{H}, \preceq \rangle$-orders, also extending the notion of (restricted) $\max$-$\max$ and $\min$-$\max$ transitivity. Subject to such results on consistency analysis, normalised additive hesitant fuzzy preference relations introduce two strategies to obtain a consensus-based model.

Then, we formally defined the generalised notion of consensus measures from $([0,1], \leq)$ to a bounded poset $\mathbb{H} = \langle \mathbb{H}, \preceq \rangle$, also studying the corresponding extensions of aggregations, implications and fuzzy negations. As one of the main contribution, the $\mathcal{CC}_{\mathcal{AI}}$- and $\mathcal{CC}_{\min,\mathcal{I}}$-consensus models are presented as new methodologies of consensus preserving main properties in the context of Typical Hesitant Fuzzy Sets, by exploring properties of admissible $\langle \mathbb{H}, \preceq \rangle$-aggregation and admissible $\langle \mathbb{H}, \preceq \rangle$-implications.

9.1 Future works

Ongoing works are focusing on other classes of implications, such as (S, N)-implications (ZANOTELLI; REISER; BEDREGAL, 2020) and including the residuation principle related to R-implications and the left-continuity of t-norms, in the context of admissible $\langle \mathbb{H}, \preceq \rangle$-orders. And, in order to show the advantage of the proposed method, further work extends case studies in cloud computing (SCHNEIDER et al., 2020), for hesitant fuzzy environments, based on the theoretical results achieved in this step of the research on admissible linear orders.

We also intend to explore new group of strategies to obtain consensus measures, mainly connected to the class of operators, satisfying commutative, nondecreasing aggregations with $\mathbf{1}_{\mathbb{H}}$-annihilator.

9.2 Publications

This section reports the publication of main results mainly connected with this thesis and its related collaborative studies with LUPS/UFPEL, FMMFCC/UFPEL and LoLITA/UFRN research groups.

9.2.1 Related publications

1. MATZENAUER, M. L.; REISER, R.; SANTOS, H.; BEDREGAL, B.; BUSTINCE,H.. Strategies on admissible total orders over typical hesitant fuzzy implications applied to decision making problems. In: International Journal of Intelligent Systems. (IJIS, 2021), p. 1-50, 2021.

2. MATZENAUER, M. L.; REISER, R.; SANTOS, H.; PINHEIRO, J.; BEDREGAL, B.. An Initial Study on Typical Hesitant (T,N)-Implication Functions. In:

18th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems. CCIS 1238, p. 747-760, 2020.

3. MATZENAUER, M. L.; REISER, R.; SANTOS, H.; BEDREGAL, B.. Typical Hesitant Fuzzy Sets: Evaluating Strategies in GDM Applying Consensus Measures. In: Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology. (EUSFLAT 2019), 2019., 2019/08. Anais. Atlantis Press, 2019/08.

4. MATZENAUER, M. L.; REISER, R.; SANTOS, H.; BEDREGAL, B.; BUSTINCE, H.. Typical Hesitant Fuzzy Implications Functions. In: Workshop Escola de Informática Teórica (WEIT2019), 2019, Passo Fundo. Anais do Workshop Escola de Informática Teórica (WEIT2019). PF: ed.UFSM, 2019. v. 1. p. 222-230.

9.2.2 Correlated publications

1. COSTA, L.; MATZENAUER, M. L. ; YAMIN, A.; REISER, R.; BEDREGAL, B.. Interval Version of Generalized Atanassov's Intuitionistic Fuzzy Index. Communications in Computer and Information Science. 1ed.: Springer International Publishing, 2018, v. 831, p. 217-229.

2. COSTA, L.; MATZENAUER, M. L.; ZANOTELLI, R.; NASCIMENTO, M.; FINGER, A.; REISER, R.; YAMIN, A.; PILLA, M.. Analysing Fuzzy Entropy via Generalized Atanassov's Intuitionistic Fuzzy Indexes. Mathware Soft Computing, v. 24, p. 22-31, 2017.

3. COSTA, L.; FINGER, A.; NASCIMENTO, M.; MATZENAUER, M. L.; ZANOTELLI, R.; REISER, R.; YAMIN, A.; PILLA, M.. Atanassov's Intuitionistic Fuzzy Entropy: Conjugation and Duality. In: Marcus Eduardo Mesquita; Graçaliz Dimuro; Regivan Santiago; Estevão Laureano. (Org.). Recent Trends on Fuzzy Systems Proc. IV CBSF. 1ed.Campinas: 2017, v. 1, p. 31-43.

4. COSTA, L.; MATZENAUER, M. L.; REISSER, R. H. S.; YAMIN, A.. Truly Intuitionistic Fuzzy Properties of Implications from Generalized Atanassov's Intuitionistic Fuzzy Index. In: Workshop Escola de Informática Teórica (WEIT2017), 2017, Santa Maria. Anais do Workshop Escola de Informática Teórica (WEIT2017). Santa Maria: ed.UFSM, 2017. v. 1. p. 278-285.

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