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Thesis

Theoretical Advances on Interval Entropy Concept Addressing a New Methodology to Multi-criteria Decision-making Problems

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Thesis presented to the Post Graduate Program in Computation of Universidade Federal de Pelotas, as a partial requirement to obtain the title of Doctor in Computer Science

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ABSTRACT

SILVA, Lidiane Costa da. **Theoretical Advances on Interval Entropy Concept Addressing a New Methodology to Multi-criteria Decision-making Problems**. 2023. 142 f. Tese (Doutorado em Computer Science) – Post Graduate Program in Computation, Center of Technological Development, Universidade Federal de Pelotas, Pelotas, 2023.

This work focuses on entropy measurements in the context of fuzzy set theory and their corresponding multi-valued logic, as it is a metric capable of quantifying not only the uncertainty of experts in data analysis but also the lack of information in systems of reasoning to which it is applied. In this research we consider the intersection of Atanassov Intuitionist Fuzzy Logic (A-IFL) and Interval-valued Fuzzy Logic (IvFL), supporting Atanassov's Interval-valued Intuitionistic Fuzzy Logic (A-IvIFL), in view of the characteristic inherent to this approach of enabling work with indeterminate and imprecise information. In this sense, we study the concept of Atanassov's Generalized Intuition Fuzzy Index (A-GIFIx), including its constructive methodology based on negation operators and automorphisms. And, from that, we built the interval version, preserving its main properties. The methodology for obtaining Entropy in A-IvIFL via aggregation of Generalized Atanassov's Interval-valued Intuitionistic Fuzzy Index (A-GIvIFIx) contributes to the analysis of decision-making systems based on multiple criteria. In the conception of the interval entropy $\omega_{\mathbf{A}}$-IvE, the notion of preservation of the interval diameters is respected, considering for its obtaining interval-valued fuzzy connectives that also have this characteristic. We use the concept of admissible orders to compare interval data. This study also contributes with a new admissible order, defined by only one injective and increasing function. This order is illustrated by a Decimal Digit Interleaving (DDI) function. Merge order promotes various method expressions for $\omega_{\mathbf{A}}$-IvE, described by compositions between negations with equilibrium functions, aggregation as means, and constrained equivalence functions. Finally, as a practical contribution, we apply the proposed methods in a decision-making problem related to video streaming traffic classification. Entropy analyzes the set of attributes, explaining the input-output relationship of data that model the FuzzyNetClass. In this computational approach for classifying traffic related to streaming video, which integrates fuzzy inference systems and machine learning algorithms, the results obtained by $\omega_{\mathbf{A}}$-IvE show promise and indicate the continuity of studies and research efforts in the area.

Keywords: atanassov's interval-valued intuitionistic fuzzy logic; intuitionistic fuzzy index; fuzzy entropy; conjugation

RESUMO

SILVA, Lidiane Costa da. **Avanços Teóricos no Conceito de Entropia Intervalar Endereçando Nova Metodologia para Problemas de Tomada de Decisão em Múltiplos Critérios**. 2023. 142 f. Tese (Doutorado em Computer Science) – Post Graduate Program in Computation, Center of Technological Development, Universidade Federal de Pelotas, Pelotas, 2023.

Este trabalho tem como foco as medidas de entropia no contexto da teoria dos conjuntos fuzzy e suas correspondentes lógicas multi-valoradas, por se tratar de uma métrica capaz de quantificar não só a incerteza dos especialistas na análise de dados como também a falta ou desconhecimento de informações em sistemas de raciocinio aos quais é aplicada. Nesta pesquisa consideramos à interseção da Lógica Fuzzy Intuicionista (A-IFL) e da Lógica Fuzzy valorada Intervalarmente (IvFL), fundamentando a Lógica Fuzzy Intuicionista de Atanassov Valorada Intervalarmente (A-IvIFL), tendo em vista a característica inerente a esta abordagem de possibilitar o trabalho com informações indeterminadas e imprecisas. Nesse sentido, estudamos o conceito de Índice Fuzzy Intuicionista Generalizado de Atanassov (A-GIFIx), incluindo sua metodologia construtiva baseada em operadores de negações e automorfismos. E, a partir disso, construímos a versão intervalar, preservando suas principais propriedades. A metodologia para obtenção da Entropia em A-IvIFL via agregação do Índice Fuzzy Intuicionista valorado Intervalarmente Generalizado de Atanassov (A-GIvIFIx) contribui para análise de sistemas de tomada de decisão baseados em multiplos critérios. Na concepção da entropia intervalar $\omega_{\mathbf{A}}$-IvE, respeita-se a noção de preservação dos diâmetros dos intervalos, considerando para sua obtenção conectivos fuzzy valorados intervalarmente que também possuam esta característica. Utizamos o conceito de ordens admissíveis para comparação de dados intervalares. Este estudo contribui ainda com uma nova ordem admissível, definida por apenas uma função injetiva e crescente. Essa ordem é ilustrada por uma função Decimal Digit Interleaving (DDI). A ordem de intercalação promove várias expressões de métodos para $\omega_{\mathbf{A}}$-IvE, descritos por composições entre negações com funções de equilíbrio, agregação como médias e funções de equivalência restrita. Por fim, como contribuição prática, aplicamos os métodos propostos em um problema de tomada de decisão relacionado à classificação de tráfego de streaming de vídeo. A entropia analisa o conjunto de atributos, explicitando a relação de entrada-saída de dados que modelam o FuzzyNetClass. Nesta abordagem computacional para classificação de tráfego relacionado a streaming de vídeo, que integra sistemas de inferência fuzzy e algoritmos de aprendizado de máquina, os resultados obtidos pelo $\omega_{\mathbf{A}}$-IvE mostram-se promissores e indicam a continuidade dos estudos e esforços de pesquisa na área.

Palavras-Chave: lógica fuzzy intuicionista valorada intervalarmente; índice fuzzy intuicionista; entropia fuzzy; conjugação

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LIST OF ABBREVIATIONS AND ACRONYMS

A-GIFIx	Generalized Atanassov’s Intuitionistic Fuzzy Index
A-GIvIFIx	Generalized Interval-valued Atanassov’s Intuitionistic Fuzzy Index
A-IFE	Atanassov‘s Intuitionistic Fuzzy Entropy
A-IFL	Atanassov‘s Intuitionistic Fuzzy Logic
A-IFS	Intuitionistic Fuzzy Set
A-IFIx	Intuitionistic Fuzzy Index
A-IvIFE	Atanassov‘s Interval-valued Intuitionistic Fuzzy Entropy
A-IvIFIx	Interval-valued Intuitionistic Fuzzy Index
A-IvIFL	Atanassov‘s Interval-valued Intuitionistic Fuzzy Logic
A-IvIFS	Interval-valued Intuitionistic Fuzzy Set
DDI	Decimal-digit Interleaving
DM	Decision Making
FA	Fuzzy Aggregation
FE	Fuzzy Entropy
FL	Fuzzy Logic
FN	Fuzzy Negation
FS	Fuzzy Set
IFN	Intuitionistic Fuzzy Negation
IvC	Interval valued Fuzzy Coimplication
IvA	Interval-valued Fuzzy Aggregation
IvFE	Interval-valued Fuzzy Entropy
IvFL	Interval-valued Fuzzy Logic
IvFN	Interval-valued Fuzzy Negation
IvFS	Interval-valued Fuzzy Set
IvI	Interval valued Fuzzy Implication
IvIFN	Interval valued Intuitionistic Fuzzy Negation

IvREF	Restricted Equivalence Interval-valued Function
IvRDF	Restricted Dissimilarity Interval-valued Function
LI	Lack of Information
LS	Lack of Specificity
MADM	Multiple Attribute Decision Making
MCDM	Multiple criteria Decision Making
OWA	Ordered Weighted Averaging Operator
SN	Standard Negation
SFN	Strong Fuzzy Negation
SIFN	Strong Intuitionistic Fuzzy Negation
SIvIFN	Strong Interval valued Intuitionistic Fuzzy Negation
T1FL	Type-1 Fuzzy Logic
T1R	Type-1 Reduction Techniques
T2FL	Type-2 Fuzzy Logic
TFS	Theory of Fuzzy Sets
TU	Type of uncertainly
ω-IvRDF	Width-preserving Interval-valued Restrict Dissimilarity Function
ω-IvREF	Width-preserving Interval-valued Restrict Equivalence Function
ω-IvE	Width-preserving Interval-valued Entropy w.r.t. the $\preceq_{XY}$-order
$\omega_{\mathbf{A}}$-IvE	Width-preserving Interval-valued Entropy w.r.t. the $\preceq_{\mathbf{A}}$-order

LIST OF SYMBOLS

χ	Universe of Discourse
$\mathcal{A}_U$	Set of Fuzzy Sets
$\mathcal{A}_{\tilde{U}}$	Set of Intuitionistic Fuzzy Sets
$\mathcal{A}_{\mathbb{U}}$	Set of Interval-valued Fuzzy Sets
$\mathcal{A}_{\tilde{\mathbb{U}}}$	Set of Interval-valued Intuitionistic Fuzzy Sets
A	Fuzzy Set
A_I	Intuitionistic Fuzzy Set
$\mathbb{A}$	Interval-valued Fuzzy Set
$\mathbb{A}_I$	Interval-valued Intuitionistic Fuzzy Set
A_C	Complement of the Fuzzy Set A
A_{I_C}	Complement of the Intuitionistic Fuzzy Set A_I
$\mathbb{A}_C$	Complement of the Interval-valued Fuzzy Set $\mathbb{A}$
$\mathbb{A}_{I_C}$	Complement of the Interval-valued Intuitionistic Fuzzy Set $\mathbb{A}I$
U	Set of Fuzzy Values
$\tilde{U}$	Set of Intuitionistic Fuzzy Values
$\mathbb{U}$	Set of Interval-valued Fuzzy Values
$\tilde{\mathbb{U}}$	Set of Interval-valued Intuitionistic Fuzzy Values
μ	Membership Degree
ν	Non-membership Degree
f	Fuzzy Function
f_N	N-dual Function
f^{ϕ}	Conjugate Function
f_I	Intuitionistic Function
f_{I_N}	Intuitionistic Dual Function
f_I^{Φ}	Intuitionistic Conjugate Function
$f_{\mathbb{U}}$	Interval-valued Function

$f_{N_{\mathbb{U}}}$	Interval-valued Dual Function
$f_{\mathbb{U}}^{\Phi_{\mathbb{U}}}$	Interval-valued Conjugate Function
$f_{\tilde{\mathbb{U}}}$	Interval-valued Intuitionistic Function
$f_{N_{\tilde{\mathbb{U}}}}$	Interval-valued Intuitionistic Dual Function
$f_{\tilde{\mathbb{U}}}^{\Phi_{\tilde{\mathbb{U}}}}$	Interval-valued Intuitionistic Conjugate Function
A_C	Complement Fuzzy Set
A_I	Intuitionistic Fuzzy Set
$\mathbb{A}$	Interval-valued Fuzzy Set
$\mathbb{A}_I$	Interval-valued Intuitionistic Fuzzy Set
$\mathbb{A}_{I_C}$	Complement Interval-valued Intuitionistic Fuzzy Set
e	Equilibrium Point
ε	Interval Equilibrium Point
N	Fuzzy Negation
N_S	Strong Standard Fuzzy Negation
N_I	Intuitionistic Fuzzy Negation
$\mathbb{N}$	Interval-valued Fuzzy Negation
$\mathbb{N}_I$	Intuitionistic Fuzzy Negation
M	Fuzzy Aggregation
$\mathbb{M}$	Interval-valued Fuzzy Aggregation
min	Minimun Operator
max	Maximun Operator
M_d	Median Operator
M_w	Weighted Mean Operator
Med	Arithmetic Mean Operator
w	Weight Parameters
σ	Ordering Permutation
T	Triangular Norm
S	Triangular Conorm
T_N	N-dual Operator Triangular Norm
S_N	N-dual Operator Triangular Conorm
T_M	Triangular Norm - Standard Intersection
S_M	Triangular Conorm - Standard Unity
T_P	Triangular Norm - Algebric Product
S_P	Triangular Conorm - Probabilistic Sum

T_D	Triangular Norm - Drastic Intersection
S_D	Triangular Conorm - Drastic Unity
T_L	Triangular Norm - Lukasiewicz Intersection
S_L	Triangular Conorm - Lukasiewicz Union
T_nM	Triangular Norm - Minimum Nilpotent
S_nM	Triangular Conorm - Maximum Nilpotent
ϕ	Fuzzy Automorphism
ψ	Automorphism Function
Φ	Interval-valued Fuzzy Automorphism
I	Fuzzy Implication
J	Fuzzy Co-Implication
I_N	Dual Fuzzy Implication
J_N	Dual Fuzzy Co-Implication
$I_{S,N}$	Fuzzy Implication
$J_{T,N}$	Fuzzy Co-Implication
I_{LK}	Łukasiewicz Implication
I_{KD}	Kleene-Dienes Implication
I_{RB}	Reichenbach Implication
I_{GR}	Gaines-Richard Implication
J_{LK}	Łukasiewicz Co-Implication
J_{KD}	Kleene-Dienes Co-Implication
J_{RB}	Reichenbach Co-Implication
J_{GR}	Gaines-Richard Co-Implication
π	Intuitionistic Fuzzy Index (IFIx)
Π	Generalized Intuitionistic Fuzzy Index (A-GIFIx)
$\tilde{\pi}$	Interval-valued Intuitionistic Fuzzy Index (A-IvIFIx)
$\tilde{\Pi}$	Generalized Interval-valued Intuitionistic Fuzzy Index (A-GIvIFIx)
E	Fuzzy Entropy (FE)
E_I	Atanassov‘s Intuitionistic Fuzzy Entropy (A-IFE)
$\mathbb{E}$	Interval-valued Fuzzy Entropy (IvFE)
$\mathbb{E}_I$	Atanassov‘s Interval-valued Intuitionistic Fuzzy Entropy (A-IvIFE)
$\mathbf{D}$	Dataset

CONTENTS

1 INTRODUCTION

Although research involving applied areas has grown in recent years, the recognized limitations of Fuzzy Logic (FL) stimulate the search for higher levels of abstraction, using extensions for the representation of information in fuzzy reasoning systems. The main concepts on Type-2 Fuzzy Logic (T2FL) were mathematically defined by Mendel and Karnik in 1998 (Karnik; Mendel, 1998), including the study of first operations on such sets.

This work considers the intersection of two relevant areas of T2FL, (i) the Intuitionistic Fuzzy Logic (A-IFL) (Atanassov, 1986) and (ii) the Interval-valued Fuzzy Logic (IvFL) (Moore, 1962). The Interval-valued Intuitionistic Fuzzy Logic (A-IvIFL) introduced by Krassimir T. Atanassov in 1969 considers the imprecision modeled by interval data as the membership function and the hesitation in determining its complementary interval-valued relation, as the non-membership functions (Atanassov, 1999).

In the A-IvIFL approach, the principles of A-IFL are preserved and the forms of data representation are expanded, adding not only the uncertainty information related to experts concerning non-necessarily complementary relations but also the imprecision information provided by the interval-valued intuitionistic fuzzy index(Reiser; Bedregal; Visintin, 2013).

Thus, we consider the study of the main properties of fuzzy entropy verified by their axiomatic concept. We considered the interval entropy of Atanossov's interval-valued intuitionistic fuzzy sets (A-IvIFE) related to the total orders proposed. The proposal is obtained by aggregation of the generalized Atanassov's intuitionistic fuzzy index (A-GIFIx) (Barrenechea et al., 2009), and corresponding constructive methodology based on fuzzy implications and involutive negations related to admissible orders are discussed.

In such context, this work introduces an interval extension of this methodology which can preserve properties by making use of dual and conjugate operators. The axiomatic concept and related constructive methodology are characterized in terms of interval-valued fuzzy implications and strong negations, preserving the main properties of an A-GIFIx. The generalized Atanassov's interval-valued intuitionistic fuzzy index and related entropy construction are proposed based on the comparison obtained by using admissible orders, in this case, focusing on Xu and Yager's admissible order (Costa et al., 2019). The new methods proposed in this research to obtain entropy are expressed by interval-valued data, resulting from the aggregation of imprecise and hesitant information. So, this work considers the concept of admissible linear orders, supporting the comparison of entropy data resulting from distinct possible input data (DE MIGUEL et al., 2016).

Focusing on the width-base interval-valued fuzzy entropy (ω_A-IvE) notion as proposed in (Takáč et al., 2019), our research is based on two relevant concepts which are described here as follows:

(i) First, consider the width of the membership intervals, which are modeled by the diameter of such interval data, as a measure of the lack of knowledge and uncertainty related to the precise membership degrees of IvFS elements. Such a concept here generates a new entropy by applying average functions;

(ii) And thus, regarding admissible orders to compare interval data and applying width-based fuzzy connectives in the data fuzzy computations. The new proposal total order requests an injective and increasing function, instanced by a Decimal Digit Interleaving (DDI) function.

The proposed width-based interval-entropy methods analyze the information not only related to the input attributes but also to the output interval-valued fuzzy sets, which result from a fuzzy inference in the fuzzy control system.

Several applications of the interval fuzzy entropy are dealing with similarity, correlation, and distance measures(Meng; Chen, 2015; Ye; Du, 2017; Saad; Abdalla; John, 2019; Tiwari, 2019). We introduce a methodology to obtain the entropy via aggregations, considering Restricted Similarity (or Dissimilarity) Interval-valued Fuzzy Function (IvREF or IvRDF) w.r.t. total orders, contributing with multi-attribute systems based on IvFL and A-IvIFL.

Finally, in this research, we illustrate the application of the proposed methods for solving a video streaming traffic classification problem.

1.1 Entropy Relevance for Interval-valued Fuzzy Sets

The relevance of entropy measures is studied in two senses, their historical and theoretical context and their application in many fields of sciences, as a special case, applied to computer science research.

Conceived as a measure of uncertainty in a fuzzy set, the fuzzy entropy measure analyzes the disorganized information, which means, the fuzziness in the fuzzy set theory. De Luca and Termini (Luca; Termini, 1972) introduced an axiomatic construction of entropy of a fuzzy set, and other preliminary approaches have been used the notion of distance measures to define fuzzy entropy and the distance from a fuzzy set and its complement (Ebanks, 1983).

The recent research of fuzzy entropy received relevant contributions from many approaches and applications of multi-valued fuzzy logic (Kabir; Papadopoulos, 2018). In (Jing; Min, 2013) and (Zhang et al., 2014), we can check entropy measures for interval-valued intuitionistic fuzzy sets (A-IvIFS), discussing their relations with similarity and inclusion measures. Also, it is possible to check on (Ji et al., 2023) a relevant research work introducing the concept of fusion information entropy.

The results in (Zhang; Zhang; Mei, 2009) promote a new axiomatic definition of entropy for interval-valued fuzzy sets (IvFS) based on distance. They investigate the relationship between entropy and the similarity measure of IvFS. In (DE MIGUEL et al., 2017), the study focuses on type-2 fuzzy entropy sets. Additionally, the research presented in (Che; Suo; Li, 2021) introduces a constructive approach within the context of A-IvIFS, exploring the properties between the distance function and the distance measure.

In (Yuan; Zheng, 2022), based on the deviation between membership and non-membership functions and the influence of hesitation, the general expression of entropy on A-IvIFS is constructed. In (Santos et al., 2019) and (Takáč et al., 2018) fuzzy entropy is obtained from fuzzy subsethood measures. And, in (Song; Wang; Xu, 2022), the hesitant entropy is performed over hesitant fuzzy sets.

Moreover, the literature shows a wide interest in the application of the entropy notions to deal with interval-valued fuzzy sets and entropy measures in medical treatment selection (Jin; Garg, 2023), image processing (segmentation cells in image thresholding), in finances (extraction and classification), in medicine (diseased cells) and in the industry (detect disconnected elements) (Al-sharhan et al., 2001).

Following a more recent contribution for classification accuracy on streaming feature selection using entropy-based uncertainty measures for fuzzy neighborhood rough sets (Xu et al., 2022). The proposed methods in this paper promote an information evaluation provided for stream video streaming traffic classification systems.

In this work, for the application of the proposal theoretical constructions for interval-valued fuzzy entropy and, of the related methods generated from these constructions, we explore the information provided by the fuzzy controller of the FuzzyNetClass approach. This hybrid approach promotes the video streaming traffic classification related to “On Demand” and “Live Streaming” video, by exploring the integration of inference systems based on interval-valued fuzzy logic and machine learning algorithms.

In this perspective, the FuzzyNetClass approach extends the related works exploring machine learning algorithms for the classification of video streaming in the flows fuzzy classifier, but preserving the specialist opinions and aspects related to its interpretability.

1.2 Principal Objective

Focusing on interval-valued entropy measures and related studies presented in (Bustince et al., 2019; Takáč et al., 2019), this proposal aims to contribute with different ways to explore the width-preserve interval-valued fuzzy entropy, offering application developers a new methodology of construction entropy measures, enabling the information analysis on interval-valued fuzzy sets. And, in addition, consider the ranking data by using total orders to compare the related information.

Based on formal studies, this work can contribute to multi-criteria decision-making problems (Ze-shui, 2007; Jin et al., 2014; Xie; Lv, 2016; Mishra; Rani, 2017), ranking the alternatives based on another way for the interval-valued intuitionistic fuzzy set model, and also offering to application developers another method of entropy construction by the intuitionistic fuzzy index, which means, obtaining A-IvIFE from A-GIvIFIx via interval-valued fuzzy implications, interval-valued idempotent aggregation and involutive negation operators.

More specifically, the following partial objectives are considered in this work.

(i) Characterization of the state-of-the-art on A-IvIFL and revision of the main concepts of A-IvIFL and A-IFL entropy measures, study definitions of basic connectives such as fuzzy implications, fuzzy negations and aggregations focusing on their algebraic properties, dual and conjugate construction based on total order defined by admissible linear order.

(ii) Revision of the axiomatic definition of A-IFIx (Bustince; Barrenechea; Mohedano, 2004) in the sense of A-GIvIFIx including concepts and main properties of representable fuzzy connectives;

(iii) Study of IvFL, main properties of interval-valued fuzzy connectives focusing on the class of representable fuzzy (co)implications based on the concept of admissible orders;

(iv) Introduction of the axiomatic definition of the A-GIvIFIx in terms of conjugated function using automorphisms and also analyzing properties of dual functions associated with the class of interval fuzzy (co)implications generated by idempotent interval aggregations;

(v) Introduction of the width-based interval fuzzy entropy notion, considering the interval data diameter as a measure of the lack of knowledge and uncertainty related to the precise membership degrees of elements in an interval-valued fuzzy set;

(vi) Generation of a new interval entropy methodology by applying width-based average functions and admissible order to compare interval data and define width-based fuzzy connectives in data fuzzy computations;

(vii) Proposal of the axiomatic definition of a constructive method to obtain interval-valued entropy; and

(viii) Exemplification and discussion related to possible applications of interval-valued entropy obtained by the constructed methodology.

1.3 Thesis Outline

The introductory chapter presents the relevance of entropy studies from the logical approaches that contextualize this research topic. Also, this chapter describes the proposals of this study and the main objectives that meet them, including this work outline at the end.

The rest of the text is organized into three main groups of chapters, starting with the preliminaries from Chapters 2 to 6. In this first part, the basic concepts of all logical approaches used in the development of this research are defined. The second group of chapters presents the main constructions developed throughout our studies and our theoretical contribution regarding entropy measures. The last group refers to the application of the methodology of interval entropy constructions, considering the 5 methods proposed and the method introduced by the case studies that were applied to network traffic classification, more specifically video streaming classes.

The Chapter 2 describes the basic concepts of Fuzzy Logic, as well as dual and conjugation operators, aggregators, disjuntive and conjuntive, implications, and coimplications functions. The axiomatic structure for the fuzzy entropy and the historic references in the area.

The basic concepts of intuitionistic fuzzy logic are studied in Chapter 3 describes some intuitionistic fuzzy connectives and order relations. In addition, Atanassov's intuitionistic fuzzy index is defined.

In Chapter 4 the generalized Atanassov's intuitionistic fuzzy index is reported, with its main axioms, obtaining dual and conjugate connectives in a special class of implications. In this chapter, we also present the concepts and properties of generalized Atanassov's intuitionistic fuzzy entropy, obtained through generalized Atanassov's intuitionistic fuzzy index considering a case study.

In Chapter 5, the main concepts of Interval-valued fuzzy logic, describing its connectives, relations of order, and in the sequence the interval intuitionistic fuzzy logic is also described from its main connectives.

The Chapter 6 presents the interval extension of generalized intuitionistic fuzzy index and relations with interval-valued fuzzy connectives.

In Chapter 7, Interval-valued Intuitionistic Fuzzy Entropy is discussed. Relationship with Intuitionistic Index and Conjugate Operators, preserving fuzzyness and intuitionistic index based on IvIFE. In addition, the injective decimal-digit interleaving $\mathbf{A}$ function and the related admissible $\preceq_{\mathbf{A}}$-order are introduced, in order to compare interval-valued fuzzy data.

Chapter 8 introduces concepts and presents the constructions proposed in this work, of width-based entropy. It exemplifies the contributions through the development of five construction methods generated by ω-preserving interval-valued fuzzy aggregations and interval-valued fuzzy equivalence function w.r.t. admissible orders.

The Chapter 9 presents the constructions referring to the Interval-valued Intuitionistic Fuzzy Entropy constructed through the aggregation of the Generalized Interval-valued Intuitionistic Fuzzy Index. As well as its dual and conjugate constructions, through negation operators and automorphisms.

In Chapter 10, we present the application of the Width-based Interval-valued Fuzzy Entropy methodology to a real network traffic classification system. The FuzzyNetClass system is described and the results are interpreted through comparison from the order constructed in this research, $\preceq_{\mathbf{A}}$-order.

In Chapter 11 the main contributions of this work, as well as the possibilities of continuity of activities.

Part I

PRELIMINARIES

2 FUZZY LOGIC

This chapter presents the concepts of Fuzzy Logic or Type-1 Fuzzy Logic (T1FL). Fuzzy Logic is an extension of classical logic which allows the representation and reasoning with uncertainty and imprecision. In this chapter, the main elements and operators of Fuzzy Logic are discussed.

A brief historical approach is presented, highlighting the development and evolution of Fuzzy Logic over time. It will discuss how this field has emerged as a powerful tool for dealing with complex and vague problems, providing a solid foundation for decision-making in uncertain situations.

The basic operators of Fuzzy Logic are presented, such as automorphisms, conjugation operators, negations, and dual operators, exploring their interrelationships. Additionally, the axiomatic definition and algebraic properties of aggregation operators such as the Ordered Weighted Averaging operator (OWA), disjunctive and conjunctive classes, are reported, as well as the notion of (co)implication operators. Examples of such operators are also discussed.

This chapter also introduces the primary notions of fuzzy entropy, proposed as a measure of uncertainty for inference information as logical support to applications based on multi-criteria decision-making and system evaluation.

Finally, we will provide the main bibliographic references, serving as resources for further study of Fuzzy Logic and its applications. Throughout this chapter, the crucial role of Fuzzy Logic as a flexible and powerful approach to deal with uncertainty and imprecision in various domains of knowledge is emphasized.

The understanding of fuzzy concepts and their practical application can contribute to the development of more robust and hybrid intelligent systems, which consider fuzzy inference as support to multi-criteria decision-making problems and machine learning techniques to achieve better performance.

However, this integration passes through the development of new methodologies, such as entropy measures, improving the evaluation and interpretability of processed information by such hybrid approaches.

2.1 Historical Approach

The Theory of Fuzzy Sets (TFS) was formalized by the mathematician Lofti Asker Zadeh (Zadeh, 1965a) by extending the concepts of the Theory of Classical Logic, characterizing the attribution of membership degrees to the elements of a fuzzy set mainly depending on application contexts. Since that technological resources based on Boolean logic were not enough to automate industrial activities or even to compute with the uncertainty of real problems (Fodor; Roubens, 1994).

The main advantage associated with the development systems based on Type-1 Fuzzy Sets (T1FS) is to obtain a mathematical model which not only is able to interpret the uncertainty of linguistic terms from natural language but also makes it possible to produce calculations even when we deal with inaccurate information in computer programming languages (Dubois; Prade, 1991).

Due to the development of countless practical possibilities and theoretical foundation allied to applications, FL is considered for uncertainty modeling in research areas such as artificial intelligence, natural language, expert systems, neural networks, control theory, and decision-making for computational processes.

2.2 Basic Concepts of Fuzzy Logic

Introduced by Zadeh in 1965 (Zadeh, 1965a), Fuzzy Logic (FL) is non-classical logic capable of numerically modeling ambiguous, uncertain, or vague information, described through a natural language aiding the modeling of the human ability to make decisions from information obtained by expert systems (Ross, 2004).

In classical set theory, an element belongs to or does not belong to a given set, however, there are cases where the pertinence between elements and sets is not precise, and it is not possible to discreetly define whether an element belongs or not to a set. Systems that model uncertainties, for example, do not always have well-defined pertinence boundaries (Siler; Buckley, 2004; Carlsson; Fuller, 2002).

In the theory of fuzzy sets, the relevance of an element to a fuzzy set is given by the related membership function. And, an element may have distinct membership degrees in each one of the fuzzy sets related to a universe of discourse $\chi \neq \emptyset$. Thus, the membership degree is a number in the unitary interval $[0, 1] = U$, obtained as the related image by the membership function defining such fuzzy set.

According with Zadeh, a fuzzy set A is characterized by its membership function $\mu_A : \chi \to U$ and $\mu_A(\mathrm{x})$ interpreting the membership degree of an element x in fuzzy set A. In this sense, a fuzzy set A can be described as a set of ordered pairs, where each generic element x in a nonempty universe χ ($\mathrm{x} \in \chi$) is associated with its degree of relevance $\mu_A(\mathrm{x})$:

$$A = \{(\mathrm{x}, \mu_A(\mathrm{x}))\colon \mathrm{x} \in \chi,\ \mu_A(\mathrm{x}) \in [0,1]\}. \tag{1}$$

In order to determine the membership functions, certain classes of functions are considered, represented by some specific algebraic properties. The most common forms are: linear by parts (triangular, trapezoidal), Gaussian, sigmoid, and singleton (unitary sets) (Ross, 2004).

By considering the natural order $(U, \leq)$, the lattice $Lat(U) = (U, \leq, \vee, \wedge, 1, 0)$ has the supremum and infimum operations both given as the following

$$x \vee y = \max(x, y) \quad \text{and} \quad x \wedge y = \min(x, y), \forall x, y \in U. \tag{2}$$

2.2.1 Automorphisms and Conjugation Operators

Automorphisms are considered a generation of new connectives, preserving as algebraic properties the classes of these logical connectives. According to (Klement; Navara, 1999, Def. 4.1), an automorphism $\phi : U \to U$ is a bijective, strictly increasing function satisfying the monotonicity property:

A1: $x \leq y$ if only if $\phi(x) \leq \phi(y)$, $\forall x, y \in U$.

In (Bustince; Burillo; Soria, 2003a), $\phi : U \to U$ is a function satisfying the continuity property and the boundary conditions:

A2: $\phi(0) = 0$ and $\phi(1) = 1$.

The set $Aut(U)$ of all automorphisms are closed under composition:

A3: $\phi \circ \phi' \in Aut(U)$, $\forall \phi, \phi' \in Aut(U)$.

In addition, there exists the inverse $\phi^{-1} \in U$, such that

A4: $\phi \circ \phi^{-1} = id_U$, $\forall \phi \in Aut(U)$.

Thus, $(Aut(U), \circ)$ is a group with the identity the function being the neutral element.

The action of an automorphism $\phi : U \to U$ on a function $f : U^n \to U$ is called the **conjugate of** f and given by the following expression:

$$f^{\phi}(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \tag{3}$$

Example 2.2.1. *For all $k, l \in \{1, .., n\}$, let $\phi_k, \psi_{k,l}$ be functions in $Aut(U)$ given by:*

$$\psi_{k,l}(x) = x^{\frac{l}{k}} \qquad \psi_{k,l}^{-1}(x) = \sqrt[l]{x^k} \tag{4}$$

$$\phi_k(x) = \frac{(kx+1)^2 - 1}{k(k+2)} \qquad \phi_k^{-1}(x) = \frac{\sqrt{(k^2+2k)x+1} - 1}{k} \tag{5}$$

Both results can be easily observed:

- *By taking $l = 1$ in Eq.(4), we obtain that $\psi_k(x) = x^k$ and $\psi_k^{-1}(x) = \sqrt[k]{x}$.*
- *And, when $k = 1$ in Eq.(5), $\phi(x) = \frac{(x+1)^2-1}{3}$ and $\phi^{-1}(x) = \sqrt{3x+1} - 1$.*

2.2.2 Fuzzy Negations and Dual Operators

A function $N : U \to U$ is a *fuzzy negation* (FN) if

N1: $N(0) = 1$ and $N(1) = 0$;

N2: If $x \geq y$ then $N(x) \leq N(y)$, $\forall\, x, y \in U$.

Fuzzy negation satisfying the involutive property below are called ***strong*** **fuzzy negations** (Bustince; Burillo; Soria, 2003a):

N3: $N(N(x)) = x$, $\forall x \in U$.

Moreover, an equilibrium point of a fuzzy negation N is a value $e \in U$ such that $N(e) = e$. And, based on (Klir; Yuan, 1995), all fuzzy negations have at most one equilibrium point meaning that, when a fuzzy negation N has an equilibrium point then it is unique.

Let N be a fuzzy negation and $f : U^n \leftrightarrow U$ be a real function. The N**-dual function of** f is denoted by $f_N : U^n \to U$ and defined as follows:

$$f_N(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n))). \tag{6}$$

Example 2.2.2. *For all $k, n \in \{1, 2 \ldots, n\}$, let $N^*, C_k : U \to U$ be strong fuzzy negations given by the corresponding expressions:*

$$N^*(x) = \frac{1-x}{1+x} \qquad C_n^k(x) = \sqrt[n-k+1]{1 - x^{n-k+1}}. \tag{7}$$

In particular, based on (Klir; Yuan, 1995, Theorem 3.4), every continuous fuzzy negation has a unique equilibrium point. Thus, the following holds:
(i) when $n = 1$ in Eq.(7), we obtain $C^k(x) = 1 - x^k$;
(ii) when $k = 1$ in Eq.(7), we have the negation $C_n(x) = \sqrt[n]{1 - x^n}$ which has $e = \sqrt[n]{\frac{1}{2}}$ as the equilibrium point, meaning that $C_n(e) = e$.
(iii) When $k = n = 1$ in Eq.(7), the standard fuzzy negation given as follows:

$$N_S(x) = 1 - x; \tag{8}$$

2.2.3 Aggregation Operators

Fuzzy set theory and aggregation operators have become powerful tools to deal with decision-making theories. Methods under fuzzy aggregation operators have been proposed and developed for effectively solving decision-making problems (Carlsson;

Fuller, 2002) and numerous theoretical results and applications have been reported in the literature.

The process of aggregation combines several numerical values into a single value that somehow represents all the others. Thus, an aggregation is a function non-decreasing, commutative, and further, preserves the boundary conditions relating to the ends of the unitary interval (Deschrijver; Kerre, 2005, Definição 4.1).

Among several definitions, see (Torra, 2005), (Calvo et al., 2002) and (Bustince; Barrenechea; Mohedano, 2004, Definition 2). An aggregation is a function $M : U^2 \to U$ demanding, for all $x, y \in U$, the following conditions:

M1: $M(\vec{0}) = M(0, 0, \ldots, 0) = 0$ and $M(\vec{1}) = M(1, 1, \ldots 1) = 1$;

M2: If $\vec{x} = (x_1, x_2, \ldots, x_n) \leq \vec{y} = (y_1, y_2, \ldots, y_n)$ then $M(\vec{x}) \leq M(\vec{y})$;

M3: $M(\overrightarrow{x_\sigma}) = M(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = M(x_1, x_2, \ldots, x_n) = M(\vec{x})$, where σ is the permutatation of the n elements in $\vec{x}$;

M4: $M(x, x, \ldots, x) = x, \forall x \in U$.

2.2.3.1 Ordered Weighted Averaging Operator (OWA)

The aggregation function *Ordered Weighted Averaging Operator* (OWA) was introduced by Yager (Yager, 1988) providing a mean of aggregating values associated with satisfying multiple criteria.

An aggregation function M is a median when $\min \leq M \leq \max$.

Thus, an OWA operator unifies both element behaviors into fuzzy sets, the conjunctive and the disjunctive.

An operator $OWA\colon U^n \to U$ is defined by the expression:

$$OWA(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} w_j x_{\sigma(j)}, \forall x_1, x_2, ..., x_n \in U, \tag{9}$$

where $\sigma\colon \{1, \ldots, n\}$, is a σ-ordering permutation with non-negatives weight-parameters w_i non-negatives verifying the following conditions:

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)} \quad \text{and} \quad \sum_{i=1}^{n} w_i = 1, \forall\ 0 \leq w_i \leq 1.$$

According to (Yager; Kacprzyk, 2012), OWA operators are aggregation functions commutative, idempotent, and have a compensatory behavior also satisfying properties **M1**, **M2** and **M3**.

Particular values of weight w_i determine parameterized families of aggregation operators which are defined from the OWA operator, including among many others, the minimun ($\min$), the maximum ($\max$), the median (M_d), the weighted mean (M_w) and the arithmetic mean (Med). See these examples expressed in accordance with Table 1.

Table 1 – Aggregations Obtained from the OWA Operator

OWA Parameters	Algebric Expression
$\begin{cases} w_i = 0 & \text{if } i \neq 1 \\ w_1 = 1 & \text{otherwise.} \end{cases}$	$\min(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \ldots \wedge x_n$
$\begin{cases} w_i = 0 & \text{if } i \neq n \\ w_n = 1 & \text{otherwise.} \end{cases}$	$\max(x_1, x_2, \ldots, x_n) = x_1 \vee \ldots \vee x_n$
$\begin{cases} w_{\frac{n+1}{2}} = 1, & \text{if n is odd;} \\ w_{\frac{n}{2}+1} = \frac{1}{2}, & \text{if n is even;} \\ w_i = 0, & \text{otherwise.} \end{cases}$	$M_d(x_1, \ldots, x_n) = \begin{cases} x_{\sigma(\frac{n+1}{2})}, & \text{if n is odd;} \\ \frac{1}{2}\left(x_{\sigma(\frac{n}{2})} + x_{\sigma(\frac{n}{2}+1)}\right), & \text{if n is even.} \end{cases}$
$\sum_{i=1}^{n} w_i = 1, \forall w_i \in \mathbb{Q}^+$	$M_{w_1,\ldots,w_n}(x_1, \ldots, x_n) = \sum_{i=1}^{n}(w_i.x_i)$
$w_i = \frac{1}{n}, \forall i.$	$Med(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i = \sum_{i=1}^{n}\left(\frac{1}{n} \cdot x_i\right)$

2.2.3.2 Disjuntive and Conjuntive Operators

An aggregation function M with n-arity $M : U^n \to U$ if $M(x_1, \ldots, x_n) \leq \min(x_1, \ldots, x_n)$. So, every conjunctive aggregation function has $\vec{0}$ as an element annihilator and if it has a neutral element this is necessarily $\vec{1}$, and M is called a semi-copula. In a dual construction, a n-arity aggregation function $M : U^n \to U$ is disjunctive if $\max(x_1, \ldots, x_n) \leq M(x_1, \ldots, x_n)$.

According to with (Klement; Navara, 1999), a triangular (co)norm is a binary aggregation $T(S) : U^2 \to U$ which is symmetric, associative, monotonic and has the neutral element. This also means that, for all $x, y, z, t \in U$, the corresponding algebraic properties are verified:

T1: $T(x, y) = T(y, x)$;
T2: $T(x, T(y, z)) = T(T(x, y), z)$;
T3: $T(x, y) \leq T(z, t)$, if $x \leq z$ and $y \leq t$
T4: $T(x, 1) = x$;

S1: $S(x, y) = S(y, x)$;
S2: $S(x, S(y, z)) = S(S(x, y), z)$;
S3: $S(x, y) \leq S(z, t)$ if $x \leq t$ and $y \leq z$;
S4: $S(x, 0) = x$

In the following, by (Klement; Mesiar; Pap, 1999), the expression of an N-dual operator of a triangular (co)norm is considered. A function $T_N(S_N) : U^2 \to U$ is a t-conorm (t-norm) if, and only if, there exists a t-norm T (t-conorm S) such that for all $x, y \in U$, the following holds:

$$T_N(x, y) = N(T(N(x), N(y))), \quad S_N(x, y) = N(S(N(x), N(y))). \tag{10}$$

A t-conorm T_N given by Eq. (10b) is called the t-conorm derived from T by the duality relation and, similarly a t-norm T_N given by Eq. (10a) is called the t-norm derived from S by the duality relation, both defined with respect to the fuzzy negation N. When N is a strong fuzzy negation, then (T, T_N) ((S, S_N)) is a pair of mutual N-dual functions.

Table 2 shows examples of pairs of mutual dual t-norms and t-conorms based on previous results from (Dubois; Prade, 2000).

Table 2 – Examples of Fuzzy Triangular (Co)Norms.

Conjuntions Disjunctions	**Algebric Expression**
Standard Intersection: Standard Unity:	$T_M(x,y) = min\ \{x,y\}$ $S_M(x,y) = max\ \{x,y\}$
Algebraic Product: Probabilistic Sum:	$T_P(x,y) = x.y$ $S_P(x,y) = x + y - xy$
Drastic Intersection:	$T_D(x,y) = \begin{cases} 0, & \text{if } x < 1, y < 1 \\ \min\{x,y\}, & \text{otherwise} \end{cases}$
Drastic Unity:	$S_D(x,y) = \begin{cases} 1, & \text{if } 0 < x \text{ e } 0 < y \\ \max\{x,y\}, & \text{otherwise} \end{cases}$
Lukasiewicz Intersection: Lukasiewicz Union:	$T_L(x,y) = \max\{x+y-1, 0\}$ $S_L(x,y) = \min\{x+y, 1\}$
Minimum Nilpotente:	$T_nM(x,y) = \begin{cases} 0, & \text{if } x + y \leq 1 \\ \min\{x,y\}, & \text{otherwise} \end{cases}$
Maximum Nilpotente:	$S_nM(x,y) = \begin{cases} 1, & \text{if } x + y \geq 1 \\ \max\{x,y\}, & \text{otherwise} \end{cases}$

2.2.4 (Co)Implications Operators

Fuzzy implications play an important role in Fuzzy Logic. In a broad sense, it is frequently applied to fuzzy control, analysis of vagueness in natural language, and techniques of soft-computing, as well as in the narrow sense, contributing to a branch of many-valued logic enabling the investigation of deep logical questions (Baczyński; Jayaram, 2007; Bustince; Burillo; Soria, 2003a; Fodor; Roubens, 1994).

A **fuzzy (co)implicator** $I(J) : U^2 \to U$ is a function verifying boundary conditions:

I0:$I(0,0) = I(0,1) = I(1,1) = 1$; **J0**: $J(0,0) = J(0,1) = J(1,1) = 0$.

Based on concepts introduced in (Fodor; Roubens, 1994), a **fuzzy (co)implication** $I(J) : U^2 \to U$ is a function verifying the following properties:

I1: If $x \leq z$ then $I(x,y) \geq I(z,y)$;
I2: If $y \leq z$ then $I(x,y) \leq I(x,z)$;
I3: $I(0,x) = 1$;
I4: $I(x,1) = 1$;
I5: $I(1,0) = 0$;

J1: If $x \leq z$ then $J(x,y) \geq J(z,y)$;
J2: If $y \leq z$ then $J(x,y) \leq J(x,z)$;
J3: $J(1,x) = 0$
J4: $J(x,0) = 0$
J5: $J(1,0) = 1$.

Several reasonable properties may be required for fuzzy (co)implications:

I6: $I(1,x) = x$;
I7: $I(x, I(y,z)) = I(y, I(x,z))$;
I8: $I(x,y) = 1 \Leftrightarrow x \leq y$;
I9: $I(x,y) = I(N(y), N(x))$, N is a SFN;
I10: $I(x,y) = 0 \Leftrightarrow x = 1$ and $y = 0$;

J6: $J(0,x) = x$;
J7: $J(x, J(y,z)) = J(y, J(x,z))$;
J8: $J(x,y) = 0 \Leftrightarrow x \geq y$;
J9: $J(x,y) = J(N(y), N(x))$, N is a SFN;
J10: $J(x,y) = 1 \Leftrightarrow x = 0$ and $y = 1$.

Main results summarized in (Baczyński; Jayaram, 2007, Lemma 2.1) provide the structure to define fuzzy negations induced by fuzzy (co)implicators. A function $I(J) : U^2 \to U$ satisfying **I0**(**J0**) and **I1**(**J1**) induces the definition of a natural fuzzy negation $N^I(N^J) : U \to U$ given as follows:

$$N^I(x) = I(x,0) \quad \text{and} \quad N^J(x) = J(x,1) \tag{11}$$

Moreover, the main results presented in (Reiser; Bedregal; Baczyński, 2013, Proposition 4.3) provide the N-dual approach for fuzzy (co)implications.

Let N be an FN and (J) I be a (co)implication. Then I_N (J_N) defined according with Eq. (6) is a (implication) coimplication given as follows

$$I_N(x,y) = N(I(N(x), N(y))), \quad J_N(x,y) = N(J(N(x), N(y))). \tag{12}$$

Let $T(S)$ be a t-(co)norm and N be a FN. An $(S,N)-$implication ($(T,N)-$coimplication) is a fuzzy (co)implication $I_{S,N} : U^2 \to U$ defined by

$$I_{S,N}(x,y) = S(N(x), y) \quad J_{T,N}(x,y) = T(N(x), y). \tag{13}$$

In this work, we also consider the class of *S-implications* which is studied in (Trillas; Valverde, 1985, Theorem 3.2) also taking into account main concepts from (Fodor; Roubens, 1994, 10, Theorem 1.13) and introduced by Baczyński and Jayaram in (Baczyński; Jayaram, 2007; Bustince; Burillo; Soria, 2003a).

Theorem 2.2.1. *(Trillas; Valverde, 1985, Theorem 3.2) Let N be a strong fuzzy negation. An implication $I : U^2 \to U$ is a* strong S-implication *if, and only if, it satisfies Properties **I1**, **I2**, **I6**, **I7**, and **I9**.*

Theorem 2.2.2. *(Baczyński; Jayaram, 2007, Theorem 1.6) Let N be a strong fuzzy* *negation. An implication $I : U^2 \to U$ is a* strong S-implication *if, and only if, it satisfies Properties **I1**, **I7** and N_I defined in Eq.(13) is a strong fuzzy negation.*

In the following, Table 3 reports the algebraic expressions of fuzzy implications considered in this work also including their corresponding fuzzy coimplications.

Each line in Table 3 is associated with a pair of mutual N_S-dual operators, meaning that Eq. (12)a and (12)b are both illustrated.

In these examples, the operators are obtained considering extensions of the Lukaziewicz, Reichenbach, Klenee-Dienes, Gaines-Richard fuzzy (co)implications in order to preserve the ordering property **I8**.

Table 3 – Fuzzy Implications, Coimplications and Duality Related.

Fuzzy Implications	**Fuzzy Coimplications**
$I_{LK}(x,y)=\begin{cases} 1, \text{if } x \leq y, \\ 1-x+y, \text{ otherwise;} \end{cases}$	$J_{LK}(x,y)=\begin{cases} 0, \text{ if } x \geq y, \\ y-x, \text{ otherwise;} \end{cases}$
$I_{KD}(x,y)=\begin{cases} 1, \text{if } x \leq y, \\ \max(1-x,y), \text{otherwise;} \end{cases}$	$J_{KD}(x,y)=\begin{cases} 0, \text{ if } x \geq y, \\ \min(1-x,y), \text{ otherwise;} \end{cases}$
$I_{RB}(x,y)=\begin{cases} 1, \text{ if } x \leq y, \\ 1-x+xy, \text{otherwise;} \end{cases}$	$J_{RB}(x,y)=\begin{cases} 0, \text{ if } x \geq y, \\ y-xy, \text{ otherwise;} \end{cases}$
$I_{GR}(x,y)=\begin{cases} 1, \text{ if } x \leq y, \\ 0, \text{otherwise;} \end{cases}$	$J_{GR}(x,y)=\begin{cases} 0, \text{ if } x \geq y, \\ 1, \text{otherwise;} \end{cases}$

2.3 Fuzzy Entropy

In 1972, De Luca and Termini (Luca; Termini, 1972) introduced an axiomatic structure for the entropy based on the concept of Shannon's entropy, in order to assess the amount of vagueness within an FS.

It is essentially a measure of fuzziness and depends on the properties demanded by fuzzy degrees. Thus, this concept measures how far the considered extension is from a set of references (which may be a crisp set).

As introduced in (Luca; Termini, 1972), a function $E : \mathcal{A}_U \to U$ is called a fuzzy entropy w.r.t. a strong fuzzy negation $N : U \to U$, which has e as the equilibrium point when the following properties are verified:

E1: $E(A) = 0$ if and only if A is crisp (non-fuzzy);

E2: $E(A) = 1$ if and only if $A = \{(\mathrm{x}, \mu_A(\mathrm{x}) = e) \colon \mathrm{x} \in \chi\}$;

E3: $E(A) \leq E(B)$ if A refines B, in the following sense: $\mu_A(\mathrm{x}_i) \leq \mu_B(\mathrm{x}_i)$ when $\mu_B(\mathrm{x}_i) \leq e$ and $\mu_A(\mathrm{x}_i) \geq \mu_B(\mathrm{x}_i)$ when $\mu_B(\mathrm{x}_i) \geq e$;

E4: $E(A) = E(A_C)$, A_C as the complement of A.

The research in new methods extending this preliminary definition of entropy measures performed over fuzzy sets have been exhaustively studied.

2.4 Main Bibliographic References

Some relevant aspects of a brief historical revision are presented at the conclusion of this chapter. The bibliographic references presented below are concerned with publication data, modeling type of uncertainty (TU) which considers the lack of information (LI-FS) and lack of specificity (LS-FS) and also provides the main characterization of entropy approaches. They are summarized in the following, see Table 4.

Table 4 – Historical Papers on Fuzzy Entropy.

Paper	**TU**	**Characterization**
(Zadeh, 1965b)	LI-FS	measuring the uncertainty information modelled by a fuzzy set
(Luca; Termini, 1972)	LI-FS	providing axiomatic definition of entropy
(Pal; Bezdek, 1984)	LI-FS	measuring fuzziness using additive and multiplicative classes
(Kosko, 1986)	LI-FS	measuring the distances from FS to crisp approach
(Liu, 1992)	LS-FS	structuring entropy as distance and similarity measures of FS
(Yager, 1998)	LS-FS	providing similarity measures and distance based on the concept of specificity

There have been several attempts to quantify the uncertainty associated with fuzzy sets as well as with Atanassov's intuitionistic fuzzy sets. Usually, such measures are called entropies. In 1965, Zadeh (Zadeh, 1965b) used entropy to measure the uncertainty modeled by a fuzzy set.

In (Luca; Termini, 1972) the definition of entropy in the setting of Fuzzy Sets Theory is introduced, by using non-probabilistic concepts in order to obtain a global measure of vagueness related to situations described by fuzzy sets. The expression of a non-probabilistic entropy measure is presented in (Kosko, 1986), conceived as a simple ratio between distances between fuzzy sets and crisp sets.

The proposal of general families of measures of fuzziness, called additive class and multiplicative class, was discussed by Pal and Bezdek (Pal; Bezdek, 1984). These measures try to quantify only one aspect of uncertainty, i.e., fuzziness. But fuzzy sets are associated with another kind of uncertainty, which is related to a lack of specificity. To make a distinction between the two, we can say that imprecision measures graduality, while specificity is related to granularity.

In (Liu, 1992), an axiomatic definition of entropy for fuzzy sets is presented, considering a discussion of distance and similarity measures of fuzzy sets, including basic relations between them. This approach enables us to quantify aspects of uncertainty, i.e., fuzziness. The concepts of σ-entropy, σ-distance measure and σ-similarity measure are also studied.

And, a discussion involving the similarity measures and distance related to the concept of specificity is proposed in (Yager, 1998), indicating their importance as a measure of uncertainty for represented information using fuzzy sets or possibility distributions. The additional results presented an extension of the specificity measure.

2.5 Summary

This chapter presented the basic concepts of fuzzy logic, through the definitions of fuzzy connectives, fuzzy negations and dual, automorphisms and conjugate functions, aggregation operators as triangle norms and conorms, also including fuzzy implications and coimplications. Finally, in addition to the definition of fuzzy entropy, a historical recapitulation of works in this area according to the approach in question is considered. We emphasize that the concepts presented in this chapter serve as a basis for the constructions developed throughout the work.

3 ATANASSOV'S INTUITIONISTIC FUZZY LOGIC

The theory of intuitionistic fuzzy sets (Atanassov; Gargov, 1989), extends the theory of fuzzy sets, associating to each element x in a universe $\mathcal{X} \neq \emptyset$, membership and non-membership degrees in an intuitionistic fuzzy set A_I, both defined in the unit interval $[0, 1]$ by the corresponding expressions ($\mu_A(\mathrm{x})$) and ($\nu_A(\mathrm{x})$), and such that the following natural relation is satisfied:

$$0 \leq \mu_{A_I}(\mathrm{x}) + \nu_{A_I}(\mathrm{x}) \leq 1. \quad (14)$$

Thus, expressions in Eq.(14) extend the fuzzy set theory, since membership and non-membership degrees are not necessarily complementary with respect to unit interval U.

3.1 Basic Concepts of Intuitionistic Fuzzy Sets

An **intuitionistic fuzzy set** (A-IFS) A_I consists into a set of pairs (μ_{A_I}, ν_{A_I}), whose components satisfy the natural restriction (Atanassov, 1986) given by Eq. (14). Therefore, it is assumed that an intuitionistic fuzzy set can be described as follows:

$$A_I = \{(\mathrm{x}, (\mu_{A_I}(\mathrm{x}), \nu_{A_I}(\mathrm{x}))) : \mathrm{x} \in \mathcal{X} \ e \ \mu_{A_I}(\mathrm{x}) + \nu_{A_I}(\mathrm{x}) \leq 1\},$$

where $\mu_{A_I}, \nu_{A_I} : \mathcal{X} \to \tilde{U}$ are the functions defining the corresponding **membership and non-membership degrees** of a element $\mathrm{x} \in \mathcal{X}$ in A_I.

As a consequence, the fuzzy set theory can be studied as a special case of intuitionistic fuzzy set theory, whose non-membership degree can be obtained through the equality: $\mu_{A_I}(\mathrm{x}) + \nu_{A_I}(\mathrm{x}) = 1$.

In the modeling system of inference rules based on A-IFL, not only the membership function $\mu_{A_I} : \chi \to U$ is considered, but also the non-membership function $\nu_{A_I} : \chi \to U$. And, each element $\mathrm{x} \in \mathcal{X} \neq \emptyset$ is associated to a membership degree $\mu_{A_I}(\mathrm{x})$ and a non-membership degree $\nu_{A_I}(\mathrm{x})$ which can be different of $N_S(\mu_{A_I}(\mathrm{x}))$, define an intuitionistic fuzzy set A_I, such that $0 \leq \mu_{A_I}(\mathrm{x}) + \nu_{A_I}(\mathrm{x}) \leq 1$.

The set of all A-IFS is denoted by $\mathcal{A}_{\tilde{U}}$.

Let $\tilde{U} = \{(x_1, x_2) \in U^2 \colon x_1 + x_2 \leq 1\}$ be the set of all intuitionistic fuzzy values which can also be defined as the set of all intuitionistic fuzzy numbers. The projection functions on $\tilde{U}$, $l_{\tilde{U}}, r_{\tilde{U}} : \tilde{U} \to U$ are given as follows:

$$l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1; \quad \text{and} \quad r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2. \tag{15}$$

And, the set of all diagonal elements is given as $\tilde{D} = \{\tilde{x} \in \tilde{U} : l_{\tilde{U}}(\tilde{x}) + r_{\tilde{U}}(\tilde{x}) = 1\}$.

According with (Atanassov; Gargov, 1998) and (Bustince; Burillo; Soria, 2003a), the usual partial order relation $\leq_{\tilde{U}}$ is given as follows:

$$(x_1, x_2) \leq_{\tilde{U}} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \tag{16}$$

for $\tilde{x}, \tilde{y} \in \tilde{U}$ such that $\tilde{0} = (0, 1) \leq_{\tilde{U}} \tilde{x}$ and $\tilde{1} = (1, 0) \geq_{\tilde{U}} \tilde{x}$, which are, respectively, the top and bottom elements of the $(\tilde{U}, \leq_{\tilde{U}})$.

Additionally, we also consider another partial order expressed as follows:

$$(x_1, x_2) \preccurlyeq_{\tilde{U}} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2. \tag{17}$$

This work studies the intuitionistic fuzzy index exploring related properties in the lattice $(\tilde{U}, \leq_{\tilde{U}}, \max_{\tilde{U}}, \min_{\tilde{U}}, \tilde{0}, \tilde{1})$ such that, for all $\tilde{x} = (x_1, x_2) \in \tilde{U}$, the following holds:

$$x \vee_{\tilde{U}} y = (\max(x, y), \min(x, y)); \quad x \wedge_{\tilde{U}} y = (\min(x, y), \max(x, y)). \tag{18}$$

Let $A_I, B_I \in \mathcal{A}_{\tilde{U}}$, the union and intersection are, respectively, given as

$$A_I \cup B_I = \{(\mathrm{x}, \max(\mu_{A_I}(\mathrm{x}), \mu_{B_I}(\mathrm{x})), \min(\nu_{A_I}(\mathrm{x}), \nu_{B_I}(\mathrm{x}))) \colon \mathrm{x} \in \chi\} \tag{19}$$

$$A_I \cap B_I = \{(\mathrm{x}, \min(\mu_{A_I}(\mathrm{x}), \mu_{B_I}(\mathrm{x})), \max(\nu_{A_I}(\mathrm{x}), \nu_{B_I}(\mathrm{x}))) \colon \mathrm{x} \in \chi\}. \tag{20}$$

And, the subsethood measure between $A_I, B_I \in \mathcal{A}_{\tilde{U}}$:

$$A_I \subseteq B_I \Leftrightarrow (\mu_{A_I}(\mathrm{x}), \nu_{A_I}(\mathrm{x})) \leq_{\tilde{U}} (\mu_{B_I}(\mathrm{x}), \nu_{B_I}(\mathrm{x})). \tag{21}$$

3.1.1 Intuitionistic Fuzzy Negations and Dual Operators

An Atanassov's intuitionistic fuzzy negation (IFN) $N_I : \tilde{U} \to \tilde{U}$ satisfies, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following properties:

$\mathbf{N}_I$ **1:** $N_I(\tilde{0}) = N_I(0, 1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1, 0) = \tilde{0}$;

$\mathbf{N}_I$ **2:** If $\tilde{x} \geq \tilde{y}$ then $N_I(\tilde{x}) \leq N_I(\tilde{y})$.

And, a strong intuitionistic fuzzy negation (SIFN) is an IFN N_I verifying the condition $\mathbf{N}_I$ **3**: $N_I(N_I(\tilde{x})) = \tilde{x}$, $\forall \tilde{x} \in \tilde{U}$.

Additionally, taking N_I as IFN, the N_I-dual function $f_{I_N} : \tilde{U}^n \to \tilde{U}$ is given by:

$$f_{I_N}(\tilde{\mathbf{x}}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))), \forall \tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n. \quad (22)$$

By (Baczyński, 2004), taking a SFN $N : U \to U$, an IFN $N_I : \tilde{U} \to \tilde{U}$ such that

$$N_I(\tilde{x}) = (N(N_S(x_2)), N_S(N(x_1))), \quad (23)$$

is also called an SIFN and **N-representable intuitionistic fuzzy negation** . Moreover, if $N = N_S$, then Eq. (23) can be reduced to $N_I(\tilde{x}) = (x_2, x_1)$.

Thus, we consider the complement of an IFS A_I which is given as

$$A_{I_C} = \{(\mathrm{x}, N(N_S(\nu_{A_I}(\mathrm{x})), N_S(N(\mu_{A_I}(\mathrm{x})))) : \mathrm{x} \in \chi, \mu_{A_I}(\mathrm{x}) + \nu_{A_I}(\mathrm{x}) \leq 1\} \subseteq \mathcal{A}_{\tilde{U}}.$$

3.1.2 Intuitionistic Fuzzy Indexes and Total orders

The intuitionistic fuzzy index (A-IFIx) of an element $\mathrm{x} \in \chi \neq \emptyset$ related to an intuitionistic fuzzy set A_I, denoted by the following expression $\pi_{A_I}(\mathrm{x})$ is named as the hesitant degree or indeterminacy degree of x in the A-IFS A_I.

According with (Xu; Yager, 2009; Atanassov, 1999), for all $\mathrm{x} \in \chi$, the intuitionistic fuzzy index of x related to A_I, is given by the function $\pi : \chi \to [0, 1]$ in the following expression:

$$\pi_{A_I}(\mathrm{x}) = 1 - \mu_{A_I}(\mathrm{x}) - \nu_{A_I}(\mathrm{x}), \text{ when } \mu_{A_I}(x) + \nu_{A_I}(\mathrm{x}) \leq 1. \quad (24)$$

Whenever $\pi_{A_I}(\mathrm{x}) = 0$, A_I is a fuzzy set A (Szmidt; Kacprzyk, 2004).

Based on the above, the accuracy and score functions $h_{A_I}, s_{A_I} : \chi \to \tilde{U}$, which are respectively given as

$$h_{A_I}(\mathrm{x}) = \mu_{A_I}(\mathrm{x}) + \nu_{A_I}(\mathrm{x}) \text{ and } s_{A_I}(\mathrm{x}) = \mu_{A_I}(\mathrm{x}) - \nu_{A_I}(\mathrm{x}) \quad (25)$$

provide the corresponding accuracy degree and the score degree of x in A_I.

So, it means that the larger $\pi_{A_I}(\mathrm{x})$ the higher the hesitancy degree of $\mathrm{x} \in \chi$. Analogously, the larger the accuracy (score) degree the smaller the hesitancy. Thus, higher scores are preferable, however, ties appear often. A tie-breaking rule in the case of equal scores uses the respective accuracy of the intuitionistic fuzzy negation. The accuracy can also be expressed as:

$$h_{A_I}(\mathrm{x}) + \pi_{A_I}(\mathrm{x}) = 1. \quad (26)$$

So, the largest $\pi_{A_I}(\mathrm{x})$ ($h_{A_I}(\mathrm{x})$), the higher the hesitancy (accuracy) degree of x in A_I.

In order to compare A-IFS by their performance, standard rules use their score and accuracy. In (Xu; Yager, 2006a) a total order on $\tilde{U}$ is proposed, enabling the comparison between two A-IFS.

Let $A_I, B_I \in \mathcal{A}_{\tilde{U}}$, we have that $A_I \preceq_{XY} B_I$ if and only if, the following holds:

$$A_I \prec_{XY} B_I \Leftrightarrow (\mu_{A_I}(\mathrm{x}), (\nu_{A_I}(\mathrm{x})) \prec_{XY} (\mu_{B_I}(\mathrm{x}), (\nu_{B_I}(\mathrm{x}))$$
$$\Leftrightarrow \begin{cases} s_{A_I}(\mu_{A_I}(\mathrm{x}), (\nu_{A_I}(\mathrm{x})) \leq s_{B_I}(\mu_{B_I}(\mathrm{x}), (\nu_{B_I}(\mathrm{x})) \text{ or} \\ s_{A_I}(x,y) = s_{A_I}(z,t) \text{ and } h_{A_I}(\mu_{A_I}(\mathrm{x}), (\nu_{A_I}(\mathrm{x})) \leq h_{A_I}(\mu_{B_I}(\mathrm{x}), (\nu_{B_I}(\mathrm{x})) \end{cases}$$
$$A_I =_{XY} B_I \Leftrightarrow \begin{cases} s_{A_I}(x,y) = s_{A_I}(\mu_{A_I}(\mathrm{x}), (\nu_{A_I}(\mathrm{x})) \text{ and} \\ h_{A_I}(\mu_{A_I}(\mathrm{x}), (\nu_{A_I}(\mathrm{x})) = h_{A_I}(\mu_{B_I}(\mathrm{x}), (\nu_{B_I}(\mathrm{x})), \forall \mathrm{x} \in \chi. \end{cases}$$

3.1.3 Intuitionistic Fuzzy Implications

Inherent properties of implications related to A-IFIx are described as follows.

Definition 3.1.1. *(Bustince; Barrenechea; Mohedano, 2004, Definition 3) An intuitionistic fuzzy implication $I_I : \tilde{U}^2 \to \tilde{U}$ is a function verifying, for all $(x,y),(x',y'),(z,t),(z',t') \in \tilde{U}$, the following properties:*

$\mathbf{I_I}0$*: If $(x,y),(z,t) \in \tilde{U}$ are such that $x+y$ = 1 and $z+t$ = 1 then $\pi((x,y),(z,t))$ = 0;*

$\mathbf{I_I}1$*: If $(x,y) \leq (x',y')$ then $I_I((x,y),(z,t)) \geq I_I((x',y'),(z,t))$;*

$\mathbf{I_I}2$*: If $(z,t) \leq (z',t')$ then $I_I((x,y),(z,t)) \leq I_I((x,y),(z',t'))$;*

$\mathbf{I_I}3$*: $I_I((0,1),(x,y)) = (1,0)$;*

$\mathbf{I_I}4$*: $I_I((x,y),(1,0)) = (1,0)$;*

$\mathbf{I_I}5$*: $I_I((1,0),(0,1)) = (0,1)$.*

Additionally, considering the A-IFIx the group of properties of fuzzy implication which are related to the hesitant index are reported in the following:

$\mathbf{I_I}6$: $\pi((x,y),(z,t)) \geq \max_{\tilde{U}}(1-x, 1-z)$;

$\mathbf{I_I}7$: If $(x,y) = (z,t)$, then $\pi((x,y),(z,t)) = \pi_{(x,y)}$;

$\mathbf{I_I}8$: If $\pi_{(x,y)} = \pi_{(z,t)}$, then $\pi((x,y),(z,t)) = \pi_{(x,y)}$.

3.1.4 Aggregation Functions on $\tilde{U}$

In the following, aggregation operators are considered in order to define intuitionistic fuzzy implications also demanding idempotent and symmetry from boundary conditions and monotonicity properties.

Proposition 3.1.1. *(Bustince; Barrenechea; Mohedano, 2004, Proposition 3) Let I be a fuzzy implication in J. Fodor's sense and let I_N be the N-dual implication of I. Let M_1, M_2, M_3, M_4 be four idempotent aggregation functions satisfying the conditions:*

$$M_1(x,y) + M_3(1-x, 1-y) \leq 1; \tag{27}$$
$$M_2(x,y) + M_4(1-x, 1-y) \geq 1, \forall x,y \in \tilde{U}. \tag{28}$$

Then $I_I : \tilde{U}^2 \to \tilde{U}$ given by the following expression:

$$I_I((x,y),(z,t)) = (I(M_1(x,1-y), M_2(z,1-t)), I_N(M_3(y,1-x), M_4(t,1-z))) \quad (29)$$

is an Atanassov's intuitionistic fuzzy implication, in sense of Fodor and Roubens (Fodor; Roubens, 1994).

The generalized intuitionistic fuzzy index extending main properties in the especial group of intuitionistic fuzzy implications, considered as follows:

Proposition 3.1.2. *(Bustince; Barrenechea; Mohedano, 2004, Corollary 1(ii)) Let $I_I : \tilde{U}^2 \to \tilde{U}$ given by Eq.(29) according with conditions of Proposition 3.1.1. The function I_I verifies the following property*

$$\pi(I_I(\tilde{x},\tilde{y})) \ \leq \ \pi\left(I(N_S(y),z), I_N(N_S(x),t)\right). \quad (30)$$

Proposition 3.1.3. *(Bustince; Barrenechea; Mohedano, 2004, Corollary 2) Let $I_I : \tilde{U}^2 \to \tilde{U}$ given by Eq.(29) according with conditions of Proposition 3.1.1. If $I_I(x,y) \geq \min(x,y)$ then I_I verifies property $\mathbf{I_{I_6}}$.*

Proposition 3.1.4. *(Bustince; Barrenechea; Mohedano, 2004) Let $I_I : \tilde{U}^2 \to \tilde{U}$ given by Eq.(29) according with conditions of Proposition 3.1.1. The function I_I verifies property $\mathbf{I_I}7$ and $\mathbf{I_I}8$.*

3.1.5 Intuitionistic Conjugation Operators

The study of automorphisms is relevant since they can be used in the generation of new connectives, preserving the main algebraic properties of classes of logical connectives (Costa; Bedregal; Neto, 2011).

Definition 3.1.2. *(Bustince; Burillo; Soria, 2003a) The function $\Phi : \tilde{U} \to \tilde{U}$ is an* ***intuitionistic automorphism*** *in $\tilde{U}$ if it is bijective and, for all $\tilde{x}, \tilde{y}$, we have to $\tilde{x} \leq_{\tilde{U}} \tilde{y}$ if and only if $\Phi(\tilde{x}) \leq_{\tilde{U}} \Phi(\tilde{y})$.*

Just as $Aut(U)$ denotes the set and all the automorphisms in U, $Aut(\tilde{U})$ indicates the set and all the intuitionistic automorphisms in $\tilde{U}$.

The action of $\Phi \in Aut(\tilde{U})$ in a function $f_I : \tilde{U}^n \to \tilde{U}$ is a function $f_I^\Phi : \tilde{U} \to \tilde{U}$, called **intuitionistic conjugated of** f_I, defined for all $\tilde{x}_1, \ldots, \tilde{x}_n \in \tilde{U}$ for expression:

$$f_I^\Phi(\tilde{x}_1, \ldots, \tilde{x}_n) = \Phi^{-1}(f_I(\Phi(\tilde{x}_1), \ldots, \Phi(\tilde{x}_n))). \quad (31)$$

According with (Costa; Bedregal; Neto, 2011, Theorem 17), let $\phi : U \to U$ be an automorphism on U. Then, for all $x \in U$, a **ϕ-representable automorphism** $\Phi : \tilde{U} \to \tilde{U}$

is defined by

$$\Phi(\tilde{x}) \;=\; (\phi(l_{\tilde{U}}(\tilde{x})), 1 - \phi(1 - r_{\tilde{U}}(\tilde{x}))). \tag{32}$$

Example 3.1.1. *Let $\phi : U \to U \in Aut(U)$ defined by $\phi_n(x) = x^n$ and let $\Phi_n : \tilde{U} \to \tilde{U}$ be a ϕ-representable automorphism given as $\Phi_n(x_1, x_2) = (x_1^n, 1 - (1 - x_2)^n)$. For instance, when $n = 2$ we have that $\Phi(\tilde{x}) = (x_1^2, 2x_2 + x_2^2)$ is a ϕ-automorphism obtained according with Eq.(32) in $Aut(\tilde{U})$.*

3.2 Summary

In this chapter, the basic concepts of intuitionistic fuzzy logic were discussed, including measures of uncertainties inherent in this approach, such as index, accuracy, and score degree. This revision also reports intuitionistic fuzzy negations and dual operators. The notion of automorphism and intuitionistic conjugation operators and their relationships are examined.

The next chapter also refers to the intuitionistic fuzzy logic approach, but it was separated to highlight the definitions of measures extended in previous studies and which serve as the theoretical fundamental for the constructions carried out in this work, mainly related to the representability of fuzzy connectives.

4 ATANASSOV'S INTUITIONISTIC FUZZY ENTROPY

As the first relevant contribution of this thesis, this chapter introduces an expression to obtain entropy in the context of Atanassov's intuitionistic fuzzy sets considering the Generalized Atanassov's Intuitionistic Fuzzy Index (A-GIFIx).

The results described in the following subsections, are presented in two research approaches. The former, studying the A-GIFIx, discusses its main properties and illustrates the algebraic construction of an A-GIFIx based on compositions between intuitionistic fuzzy implications and negations. And the latter is to obtain Atanassov's intuitionistic fuzzy entropy based on an A-GIFIx.

4.1 Generalized Atanassov's Intuitionistic Fuzzy Index

In (Bustince et al., 2011), the concept of the Generalized Atanassov's Intuitionistic Fuzzy Index (A-GIFIx) is characterized in terms of fuzzy implication operators. In addition, a constructive method with automorphisms is also proposed in (Barrenechea et al., 2009), together with some special properties of A-GIFIx (DA SILVA et al., 2016).

In this section, we study the properties of A-GIFIx, contributing with an incremental study of its duality and conjugation analysis (DA SILVA et al., 2016).

Definition 4.1.1. *(Bustince et al., 2011, Definition 1) A function* $\Pi : \tilde{U} \rightarrow U$ *is called a generalized intuitionistic fuzzy index associated with a strong negation* N_I *if, for all* $x_1, x_2, y_1, y_2 \in U$*, it holds that:*

$\mathbf{\Pi 1}$*:* $\Pi(x_1, x_2) = 1$ *if and only if* $x_1 = x_2 = 0$*;*

$\mathbf{\Pi 2}$*:* $\Pi(x_1, x_2) = 0$ *if and only if* $x_1 + x_2 = 1$*;*

$\mathbf{\Pi 3}$*:* $(y_1, y_2) \preccurlyeq_{\tilde{U}} (x_1, x_2)$ *implies* $\Pi(x_1, x_2) \leq \Pi(y_1, y_2)$

$\mathbf{\Pi 4}$*:* $\Pi(x_1, x_2) = \Pi(N_I(x_1, x_2))$ *when* N_I *is a SIFN.*

In particular, the following interpretations are held:

(i) Property $\Pi 1$ states the lack of information should be maximum, whenever there exists no information supporting/against a proposition;

(ii) In contrast, Property $\Pi 2$ states that when the membership and non-membership degrees are exactly complementary (related to fuzzy sets), the lack of information is minimal.

(iii) By $\Pi 3$, when the membership and the non-membership values increase, the lack of information decreases since the considered A-IFS is closer to being an FS.

(iv) And analyzing Property $\Pi 4$, no new information or knowledge is obtained by negation. With respect to the use of the $\preccurlyeq_{\tilde{U}}$ in ordering instead of $\leq_{\tilde{U}}$, one can observe that the A-GIFIx is neither increasing nor decreasing with respect to $\leq_{\tilde{U}}$, whereas it is decreasing with respect to $\preccurlyeq_{\tilde{U}}$.

4.1.1 Obtaining A-GIFIx based on Fuzzy (Co)Implications

A constructive method to obtain an A-GIFIx based on fuzzy (co)implications are proposed in (Bustince et al., 2011) and reported below:

Proposition 4.1.1. *(Bustince et al., 2011, Theorem 3) Let N be a strong negation SFN. A function $\Pi : \tilde{U} \to U$ is a A-GIFIx if only if there exists a function $I : U^2 \to U$ verifying $\mathbf{I1}$, $\mathbf{I8}$, $\mathbf{I9}$ and $\mathbf{I10}$ such that*

$$\Pi(\tilde{x}) \quad = \quad N(I(N_S(x_2), x_1)), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{33}$$

The N_S-dual construction related to Proposition 4.1.1 is considered in the following:

Proposition 4.1.2. *(DA SILVA et al., 2016, Proposition 1) Let N_I be an N-representable IFN obtained by a SFN N. A function $\Pi : \tilde{U} \to U$ is a A-GIFIx if only if there exists a function $J : U^2 \to U$ verifying $\mathbf{J1}$, $\mathbf{J8}$, $\mathbf{J9}$ and $\mathbf{J10}$ such that*

$$\Pi_J(\tilde{x}) \quad = \quad J(N(1 - x_2), N(x_1)), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{34}$$

Theorem 4.1.1. *Based on conditions of Propositions 4.1.1 and 4.1.2 , when $I, J : U \to U$ is a pair of mutual N-dual fuzzy implications, meaning that $I_N = J$ or $J_N = I$, the following holds:*

$$\Pi(\tilde{x}) \quad = \quad \Pi_J(\tilde{x}), \ \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{35}$$

Proof. By Proposition 4.1.1, we have that:

$$\Pi(\tilde{x}) \quad = \quad N(I(N_S(y), x)) = N(J_N(N_S(y), x)) = J(N(N_S(y), N(x)) = \Pi_J(\tilde{x})$$

Therefore, Theorem 4.1.1 is also verified. □

4.1.2 Duality Relation of the Intuitionistic Fuzzy Index

Now, the duality and conjugation properties related to A-GIFIx are presented.

Theorem 4.1.2. *Let N be a SFN and N_I be its corresponding N-representable SIFN. For an A-GIFIx $\Pi : \tilde{U} \to U$ the following holds:*

$$\Pi_N(\tilde{x}) = N(\Pi(\tilde{x})), \forall \tilde{x} \in \tilde{U}. \quad (36)$$

Proof. By Eq.(23) and Property $\mathbf{\Pi 4}$, $\Pi_N(\tilde{x}) = N(\Pi(N_I(\tilde{x}))) = N\left(\Pi(\tilde{x})\right)$. □

Results in Proposition 4.1.3 are related to (DA SILVA et al., 2016, Proposition 2).

Proposition 4.1.3. *(DA SILVA et al., 2018, Proposition 3) Let N be a SFN and N_I be its corresponding N-representable SIFN. For a A-GIFIx(N) $\Pi(\Pi_J) : \tilde{U} \to U$ the following holds:*

$$\left(\Pi\right)_N\left(\tilde{x}\right) = I(N_S(x_2), x_1); \quad (37)$$

$$\left(\Pi_J\right)_N\left(\tilde{x}\right) = N(J(x_1, N_S(x_2))), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \quad (38)$$

Proof. For all $\tilde{x} \in \tilde{U}$, $\left(\Pi_I\right)_N\left(\tilde{x}\right) = N\left(\Pi_I(N_I(\tilde{x}))\right) = N\left(\Pi(\tilde{x})\right) = I(N_S(x_2), x_1)$. Its dual construction can be proved analogously. □

In diagrams of Figures 1 and 2 the following denotation is considered:
(i) $C(I)$ and $C(J)$ denote the classes of fuzzy implications and coimplications verifying the conditions in Propositions 4.1.1 and 4.1.2;
(ii) $C(N_I)$ denotes the class of strong fuzzy negations on $\tilde{U}$;
(iii) $C(\Pi)$ provides denotation to the class of all A-GIFIx.
These interrelations summarize the results stated in Propositions 4.1.1 and 4.1.2, Theorem 4.1.2 and Proposition 4.1.3.

The figures 1 and 2 represents the possibility of obtaining the A-GIFIX from an (co)implication function and a strong fuzzy negation. It is observed that, according to the functions available, it is possible to reach the same result by applying those indicated by the arrows in the diagram.

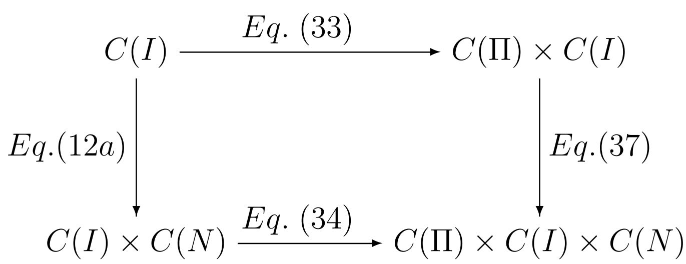

Figure 1 – A-GIFIx Obtained by Fuzzy Implications and Corresponding Dual Operator.

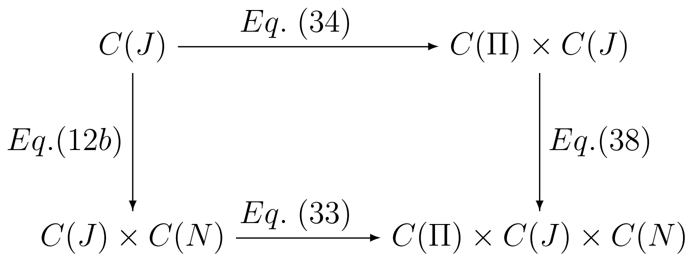

Figure 2 – A-GIFIx Obtained by Fuzzy Coimplications and Corresponding Dual Operator.

4.1.3 Hesitant and Accuracy Related to Intuitionistic Fuzzy Index

Consequently, one can describe hesitance and accuracy in terms of A-GIFIx. See, in (DA SILVA et al., 2016, Corollary 1), it is shown that the A-IFIx $\pi : \tilde{U} \to U$, can be defined as an A-GIFIx by considering the Lukaziewicz fuzzy implication $I_{LK} : U \to U$ given by the following expression

$$\Pi_{I_{LK}}(\tilde{x}) = \pi(\tilde{x}) = 1 - \mu_A(\mathrm{x}) - \nu_A(\mathrm{x})), \forall \mathrm{x} \in \chi; \tag{39}$$

and analogously, its N_S-dual construction can be given as follows:

$$(\Pi_{I_{LK}})_{N_S}(\tilde{x}) = h(\tilde{x}) = \mu_A(\mathrm{x}) + \nu_A(\mathrm{x})), \forall \mathrm{x} \in \chi. \tag{40}$$

Table 5 does not only illustrate Proposition 4.1.1 and Proposition 4.1.2, but also presents additional examples of A-GIFIx associated with the following fuzzy implications: Lukaziewicz (LK), Klenee-Dienes (KD), Reichenbach (RB) and Gaines-Rescher (GR).

4.1.4 Conjugation related to A-GIFIx

In the following, we study the action of automorphisms in A-GIFIx obtained by fuzzy (co)implications.

Proposition 4.1.4. *(DA SILVA et al., 2016, Prop. 4) Let $\Phi \in Aut(\tilde{U})$ be a ϕ-representable automorphism, $N^\phi : U \to U$ be the ϕ-conjugate of a SFN N. A function $\Pi^\Phi : \tilde{U} \to U$ is a A-GIFIx(N_I^ϕ) given by*

$$\Pi^\Phi(x_1, x_2) \;=\; (\phi^{-1}(\Pi(\phi(x_1)), 1 - \phi(1 - x_2)), \tag{41}$$

whenever $\Pi : \tilde{U} \to \tilde{U}$ is also a A-GIFIx.

Proposition 4.1.5. *(DA SILVA et al., 2018, Proposition 5) Let $\phi \in Aut(U)$ be an automorphism, $N^\phi : U \to U$ be a ϕ-conjugate of a SFN $N : U \to U$ and $I^\phi : U^2 \to U$ be a*

Table 5 – Generalized Intuitionistic Fuzzy Index Associated with the Standard Negation.

Pairs of Dual Fuzzy (Co)Implications	Dual A-GIFIx
$I_{LK}(x,y)=\begin{cases}1, \text{if } x \leq y,\\ 1-x+y, \text{otherwise};\end{cases}$ $J_{LK}(x,y)=\begin{cases}0, \text{if } x \geq y,\\ y-x, \text{otherwise};\end{cases}$	$\Pi_{LK}(x,y)=1-x-y$ $(\Pi_{LK})_{N_{SI}}(x,y)=x+y$
$I_{KD}(x,y)=\begin{cases}1, \text{if } x \leq y,\\ \max(1-x,y), \text{ otherwise};\end{cases}$ $J_{KD}(x,y)=\begin{cases}0, \text{if } x \geq y,\\ \min(1-x,y), \text{otherwise};\end{cases}$	$\Pi_{KD}(x,y)=1-\max(x,y)$ $(\Pi_{KD})_{N_{SI}}(x,y)=\max(x,y)$
$I_{RB}(x,y)=\begin{cases}1, \text{if } x \leq y,\\ 1-x+xy, \text{otherwise};\end{cases}$ $J_{RB}(x,y)=\begin{cases}0, \text{if } x \geq y,\\ 1-x-y+xy \text{ otherwise};\end{cases}$	$\Pi_{RB}(x,y)=1-x-y+xy$ $(\Pi_{RB})_{N_{SI}}(x,y)=y-xy$
$I_{GR}(x,y)=\begin{cases}1, \text{if } x \leq y,\\ 0, \text{otherwise};\end{cases}$ $J_{GR}(x,y)=\begin{cases}0, \text{if } x \geq y,\\ 1, \text{otherwise};\end{cases}$	$\Pi_{GR}(x,y)=1$ $(\Pi_{GR})_{N_{SI}}(x,y)=0$

ϕ-conjugate of $I : U^2 \rightarrow U$. A function $\Pi_{I^\phi}(\Pi_{J^\phi}) : \tilde{U} \rightarrow U$ given by

$$\Pi_{I^\phi}(x_1,x_2) = N^\phi(I^\phi(1-x_2,x_1)), \quad (42)$$

$$\Pi_{J^\phi}(x_1,x_2) = J^\phi(N^\phi(1-x_2),N^\phi(x_1)), \quad (43)$$

is a A-GIFIx(N) whenever $\Pi_I(\Pi_J) : \tilde{U} \rightarrow \tilde{U}$ is also a A-GIFIx(N).

The main results formalized in Propositions 4.1.1 and 4.1.2 together with Propositions 4.1.4 and 4.1.5 are summarized in the commutative diagrams in Figures 3 and 4, respectively.

In diagrams of Figures 3 and 4 the following denotation is considered:
(i) $C(I)$ and $C(J)$ denotes the classes of fuzzy implications and coimplications
(ii) $Aut(U)$ denotes the class of automorphisms on U;
(iii) $C(\Pi)$ provides denotation to the class of all A-GIFIx.
These interrelations summarize the results stated in Propositions 4.1.4 and 4.1.5.

Moreover, Table 6 illustrates the construction of A-GIFIx obtained by the ϕ-conjugate implications as described in Table 5. In these examples, the conjugation of the described implications, Lukaziewicz (LK), Klenee-Dienes (KD), Reichenbach (RB), and Gaines-Rescher (GR), which are generated by the following automorphisms:

$$\phi(x) = x^2 \; and \; \phi^{-1} = \sqrt{x}.$$

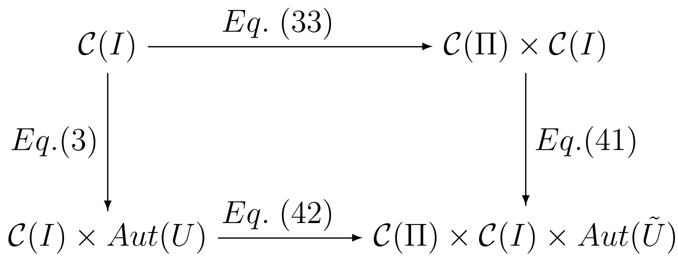

Figure 3 – A-GIFIx Obtained by Fuzzy Implications and Conjugate Operator.

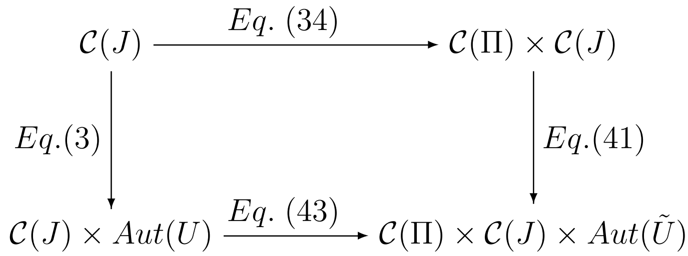

Figure 4 – A-GIFIx Obtained by Fuzzy Complications and Conjugate Operator.

Table 6 – A-GIFIx Obtained by Conjugate Functions

Fuzzy Implications	A-GIFIx
$I^{\phi}_{KD}(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{\max(1-x^2, y^2)}, \text{otherwise}; \end{cases}$ $J^{\phi}_{KD}(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{\min((1-x)^2, y^2)}, \text{otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{KD}}(x,y) = 1 - \sqrt{\max(x^2, 1-(1-y)^2)}$ $\Pi_{J^{\phi}_{KD}}(x,y) = \sqrt{1 - \max(x^2, 1-(1-y)^2)})$
$I^{\phi}_{LK}(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1-x^2+y^2}, \text{ otherwise}; \end{cases}$ $J^{\phi}_{LK}(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{1-x^2+y^2}, \text{ otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{LK}}(x,y) = 1 - \sqrt{1+x^2-(1-y)^2}$ $\Pi_{J^{\phi}_{LK}}(x,y) = \sqrt{x^2-(1-y)^2}$
$I^{\phi}_{RH}(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1-x^2+x^2y^2}, \text{otherwise}; \end{cases}$ $J^{\phi}_{RH}(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{y^2-x^2y^2}, \text{otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{RH}}(x,y) = 1 - \sqrt{1-(1-y)^2(1-x^2)}$ $\Pi_{J^{\phi}_{RH}}(x,y) = \sqrt{(1-y)^2(1-x^2)}$
$I^{\phi}_{GR}(x,y) = \begin{cases} 1, \text{if} x \leq y, \\ 0, \text{otherwise}; \end{cases}$ $J^{\phi}_{GR}(x,y) = \begin{cases} 0, \text{if} x \geq y, \\ 1, \text{otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{GR}}(x,y) = 0$ $\Pi_{J^{\phi}_{GR}}(x,y) = 1$

Now, we attend to the class of (S,N)-implications, which is a class of explicit representable fuzzy implications, from compositions of t-conorms and fuzzy negations, discussing their main properties and duality relations.

4.1.5 Obtaining (S,N)-Implications and (T,N)-Coimplications via A-GIFIx

In the following, extending results from (DA SILVA et al., 2016), the classes of (S,N)-implications and (T,N)-coimplications are considered in order to obtain new expressions of A-GIFIx.

Proposition 4.1.6. *(DA SILVA et al., 2018, Proposition 6) Let N be a SFN. A function $\Pi : \tilde{U} \to U$ is an A-GIFIx related to a SFN $N : U \to U$ iff there exists an S-implication (T-coimplication) $I_S(J_T) : U^2 \to U$ such that the following holds:*

$$\Pi_{I_{S,N}}(x_1, x_2) = S(N_S(x_2), N(x_1))); \quad (44)$$
$$\Pi_{J_{T,N}}(x_1, x_2) = T(N_S(x_2), N(x_1)). \quad (45)$$

In the following, by considering the standard negation N_S, it is possible to obtain an A-GIFIx making use of t-(co)norms:

Corollary 4.1.1. *When $N = N_S$, the A-IFIx can be expressed as*

$$\Pi_{I_{S,N_S}}(x_1, x_2) = N_S(S_{N_S}(x_1, x_2)); \quad (46)$$
$$\Pi_{J_{T,N_S}}(x_1, x_2) = N_S(T_{N_S}(x_1, x_2)). \quad (47)$$

Proof. Strainghtforward Proposition 4.1.6. □

4.2 Intuitionistic Fuzzy Entropy

In this section, the study of Atanassov's intuitionistic fuzzy entropy follows from results stated in (Bustince et al., 2011). Such an approach focuses more on the degree of intuitionism of an A-IFS than the fuzziness of an intuitionistic fuzzy set.

Definition 4.2.1. *(Bustince et al., 2011, Definition 2) A real function $E_I : \mathcal{A}_I \to U$ is called an Atanassov's intuitionistic fuzzy entropy (A-IFE) if the following properties are verified:*

E_I**1:** $E_I(A_I) = 0$ *if and only if* $\mathcal{A}_{\tilde{U}} \in \mathcal{A}$,

E_I**2:** $E_I(A_I) = 1$ *if and only if* $\mu_A(\mathrm{x}) = \nu_A(\mathrm{x}) = 0$, $\forall \mathrm{x} \in \chi$,

E_I**3:** $E_I(A_I) = E(A_{I_C})$,

E_I**4:** *if* $A_I \preccurlyeq_{\tilde{U}} B_I$ *then* $E_I(A_I) \geq E_I(B_I), \forall A_I, B_I \in \mathcal{A}_{\tilde{U}}$.

According with (Bustince et al., 2011), some interpretations of Definition 4.2.1:

(i) By **E**$_I 1$, the lack of information should be the maximum whenever there is no information supporting a proposition, and if there is no information against the same proposition;

(ii) In the opposite position, by **E**$_I 2$, the lack of information;

(iii) As the third axiom, **E**$_I 3$ states that the lack of information decreases if the membership and the non-membership values increase meaning that such A-IFS is closer to being a fuzzy set;

(iv) And, by the last axiom, $\mathbf{E}_I4$ we have that the order relation when entropy is applied to intervals will be inverse to the order relation of intervals.

Proposition 4.2.1. *(DA SILVA et al., 2018, Proposition 10) Let Φ be a ϕ-representable automorphism in $Aut(\tilde{U})$ and $E_I : \mathcal{A}_{\tilde{U}} \to U$ be an A-IFE. Then, for all $A_I \in \mathcal{A}_I$, the Φ-conjugate function $E_I^{\Phi} : \mathcal{A}_I \to U$ is also an A-IFE.*

Properties related to A-IFE obtained by aggregation of A-GIFIx are discussed below by considering a finite set $\chi = \{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$.

Proposition 4.2.2. *(Bustince et al., 2011, Prop. 4) Let M be an aggregation on U, N be a SFN, Π be an A-GIFIx. Then, for all $A_I \in \mathcal{A}_{\tilde{U}}$, the mappings $E_I : \mathcal{A}_{\tilde{U}} \to U$ defines an Atanassov's intuitionistic fuzzy entropy (A-IFE), respectively expressed by*

$$E_I(A_I) \quad = \quad M_{i=1}^{n}\Pi(A_I(\mathrm{x}_i)), \forall \mathrm{x}_i \in \chi. \tag{48}$$

Proposition 4.2.3. *(DA SILVA et al., 2018, Proposition 10) Let M be an aggregation on U, N be a SFN, Π be an A-GIFIx(N) and $\phi \in Aut(U)$. Then, for all $A_I \in \mathcal{A}_{\tilde{U}}$, the mappings $E^{\Phi} : \mathcal{A}_{\tilde{U}} \to U$ expressed by*

$$E_I^{\Phi}(A_I) \quad = \quad M_{i=1}^{n}\Pi^{\Phi}(A_I(\mathrm{x}_i)), \forall \mathrm{x}_i \in \chi, \tag{49}$$

defines Atanassov's intuitionistic fuzzy entropy.

Let $C(E_I)$ be the class of all A-IFE. The diagram below summarizes the main results related to the classes of A-GIFIx and A-IFE.

The main results in Propositions 4.2.2 and 4.2.3 together with Propositions 4.1.4 and 4.1.5 are summarized in the diagram below (Figure 5):

In diagram of Figure 5 the following denotation is considered:
(i) $C(\Pi)$ provides denotation to the class of all A-IFIx;
(ii) $Aut(U)$ denotes the class of all automorphisms on U;
(iii) $C(E_I)$ provides denotation to the class of entropy related to an intuitionistic fuzzy set.
These interrelations summarize the results stated in Propositions 4.1.4 and 4.1.5.

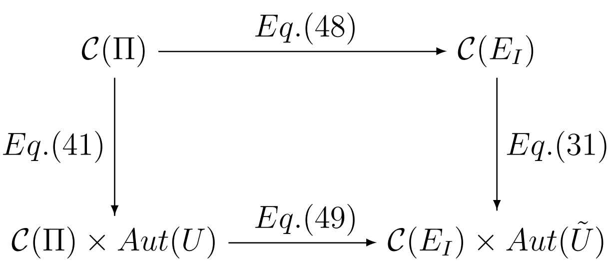

Figure 5 – Relatioship Between A-GIFIx(N) and A-IFE Aut($\tilde{U}$)

In the following, an A-IFE is obtained from A-GIFIx as conceived in (Bustince; Burillo; Soria, 2003a), with respect to its dual and conjugate constructions.

The next two propositions report the main results from (DA SILVA et al., 2016) and (Bustince et al., 2011).

Proposition 4.2.4. *Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $M : U^n \to U$ be an aggregation function and $I_N : U^2 \to U$ be a N-dual operator of an implication $I : U^2 \to U$ which satisfies properties $\mathbf{I_I1}$, $\mathbf{I_I8}$, $\mathbf{I_I9}$ and $\mathbf{I_I10}$, as discussed in Proposition 4.1.1. Then, for all $A_I \in \mathcal{A}_{\tilde{U}}$, the mappings $E_{I(N,I)}, E_{I(N^\Phi,I^\Phi)} : \mathcal{A}_{\tilde{U}} \to U$ defined by*

$$E_{I(N,I)}(A_I) = M_{i=1}^n N(I(1-\nu_A(\mathrm{x}_i), \mu_A(\mathrm{x}_i))), \quad (50)$$

$$E_{I(N^\Phi,I^\Phi)}(A_I) = M_{i=1}^n N^\phi(I^\phi(1-\nu_A(\mathrm{x}_i), \mu_A(\mathrm{x}_i))), \forall \mathrm{x}_i \in \chi, \quad (51)$$

provide new expressions of A-IFE obtained from an A-GIFIx.

Proof. Straightforward from Propositions 4.1.2 and 6.3.3, also taking Eq.(33) and (139). □

Proposition 4.2.5. *(DA SILVA et al., 2018, Proposition 12) Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $M : U^n \to U$ be an aggregation function and $J_N : U^2 \to U$ be a N-dual operator of a coimplication $J : U^2 \to U$ satisfying properties $\mathbf{J2}$, $\mathbf{J8}$, $\mathbf{J9}$ and $\mathbf{J10}$, according with Proposition 4.1.2. Then, for all $A_I \in \mathcal{A}_{\tilde{U}}$, the mappings $E_{I(J,N)}, E_{I(J^\phi,N^\phi)} : \mathcal{A}_{\tilde{U}} \to U$ defined by*

$$E_{I(J,N)}(A_I) = M_{i=1}^n J\left(N(1-\nu_A(\mathrm{x}_i)), N(\mu_A(\mathrm{x}_i))\right), \quad (52)$$

$$E_{I(J^\phi,N^\phi)}(A_I) = M_{i=1}^n J^\phi\left(N^\phi(1-\nu_A(\mathrm{x}_i)), N^\phi(\mu_A(\mathrm{x}_i))\right), \forall x_i \in \chi, \quad (53)$$

are also Atanassov's intuitionistic fuzzy entropy (A-IFE).

Proposition 4.2.6. *Let $E_J, E_{J_N} : \mathcal{A}_{\tilde{U}} \to U$ be A-IFE according with Propositions 4.2.4 and 4.2.5. Then, for all $A_I \in \mathcal{A}_{\tilde{U}}$, the following holds:*

$$E_{I(J_N,N)}(A_I) = E_{I(J,N)}(A_I) \text{ and } E_{I(I_N,N)}(A_I) = E_{I(I,N)}(A_I) \quad (54)$$

4.3 Illustrating A-IFE based on A-GIFIx

Among the several applications of A-IFIx such as similarity, correlation, and distance measures, we report a methodology to obtain the entropy via A-IFIx, contributing with multi-attribute systems based on IFL (Lin; Xia, 2006).

In order to illustrate and compare the above proposed method to obtain A-IFE making using aggregation of A-GIFIx, six expressions of A-IFEs introduced in (Liu; Ren, 2014a) are considered. For $A_I = \{(\mathrm{x}_i, \mu_(\mathrm{x}_i), \nu_(\mathrm{x}_i)) \colon \mathrm{x}_i \in$

$\chi\}$, see their references and related algebraic expressions listed below:

1.(Qian-sheng; Jiang, 2008) $E_1(A_I) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{2}(2\mu_A(\mathrm{x}_i) + \Pi_A(\mathrm{x}_i)) \cdot \log_2 \frac{1}{2}(2\mu_A(\mathrm{x}_i)+ +\Pi_A(\mathrm{x}_i)) + \frac{1}{2}(2\nu_A(\mathrm{x}_i) + \Pi_A(\mathrm{x}_i)) \cdot \log_2 \frac{1}{2}(2\nu_A(\mathrm{x}_i) + \Pi_A(\mathrm{x}_i))\right]$

2. (Ye, 2010) $E_2(A_I) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\sqrt{2}cos(\mu_A(\mathrm{x}_i) - \nu_A(\mathrm{x}_i))\frac{\pi}{4} - 1\right)\frac{1}{\sqrt{2}-1}\right]$

3. (Verma; Sharma, 2013) $E_3(A_I) = \frac{1}{2n(\sqrt{e}-1)}\sum_{i=1}^{n}\left[2\mu_A(\mathrm{x}_i) + \pi_A(\mathrm{x}_i)) \cdot e^{1-\frac{1}{2}(2\mu_A(\mathrm{x}_i)+\pi_A(\mathrm{x}_i))} + \frac{1}{2}(2\nu_A(\mathrm{x}_i) + \pi_A(\mathrm{x}_i))e^{1-\frac{1}{2}(2\nu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))} - 1\right]$

4. (Wei; Gao; Guo, 2012) $E_4(A_I) = \frac{1}{n}\sum_{i=1}^{n} cos\left(\frac{\mu_A(\mathrm{x}_i)-\nu_A(\mathrm{x}_i)}{(1+\pi_A(x_i))}\frac{\pi}{4}\right)$

5. (Yue; Jia; Ye, 2009) $E_5(A_I) = \frac{1}{n}\sum_{i=1}^{n} cot\left(\frac{\pi}{4} + \frac{|\mu_A(\mathrm{x}_i)-\nu_A(\mathrm{x}_i)|}{(1+\pi_A(\mathrm{x}_i))} * \pi\right)$

6. (Liu; Ren, 2014a) $E_6(A_I) = \frac{1}{n}\sum_{i=1}^{n} cot\left(\frac{\pi}{4} + (|\mu_A(\mathrm{x}_i) - \nu_A(\mathrm{x}_i)| * (1 - \pi_A(\mathrm{x}_i)))\frac{\pi}{4}\right)$

Let an A-IFE obtained from Eq.(33)a by taking the arithmetic mean ($M = AM$). Thus, for $A_I = \{(\mathrm{x}_i, \mu_(\mathrm{x}_i), \nu_(\mathrm{x}_i) \colon \mathrm{x}_i \in \chi\}$, it is given as follows:

$$E_7\,(A_I) = \frac{1}{n}\sum_{i=1}^{n}\Pi_I(\mathrm{x}_i), \forall \mathrm{x} \in \chi.$$

Let n be a positive integer and $\chi = \{x_1, x_2, \ldots, x_n\}$ be a finite set in order to define the comparable A-IFS A^n, which are given by the following expression:

$$A_I^n = \{(\mathrm{x}_i, (\mu_A(\mathrm{x}_i)^n, 1 - [1 - \nu_A(\mathrm{x}_i)]^n) : \mathrm{x}_i \in \chi\}. \tag{55}$$

For the finite universe $\chi = \{6, 7, 8, 9, 10\}$ and the A-IFS explicitly given as $A_I = \{(6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.6, 0.2), (9, 0.9, 0.0), (10, 1.0, 0.0)\}$, it results on the following intuitionistic fuzzy sets: :

$A_I^{\frac{1}{2}} = \{(6, 0.3162, 0.5527), (7, 0.5477, 0.2929), (8, 0.7746, 0.1056), (9, 0.9487, 0.0), (10, 1.0, 0.0)\}$;

$A_I^1 = \{(6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.6, 0.2), (9, 0.9, 0.0), (10, 1.0, 0.0)\}$;

$A_I^2 = \{(6, 0.01, 0.96), (7, 0.09, 0.75), (8, 0.36, 0.36), (9, 0.81, 0.0), (10, 1.0, 0.0)\}$;

$A_I^3 = \{(6, 0.001, 0.992), (7, 0.027, 0.875), (8, 0.216, 0.488), (9, 0.729, 0.0), (10, 1.0, 0.0)\}$;

$A_I^4 = \{(6, 0.0001, 0.9984), (7, 0.0081, 0.9375), (8, 0.1296, 0.5904), (9, 0.6561, 0.0), (10, 1.0, 0.0)\}$.

Based on Eq.(55), the characterization of linguistic variables can be presented as:

$A_I^{1/2}$:"rather large" A_I: "quite large" A_I^2:"large" A^3:"very large" A_I^4:"extremely large"

As a remark, since the above presented A-IFS from $A^{\frac{1}{2}}$ to A^4 are defined as a comparable structure by the usual order on $\mathcal{A}_{\tilde{U}}$, from axioms of the logical approach defining the A-GIFIx, the entropy related to these IFS follow the next ordering:

$$A_I^{1/2} \leq_{\tilde{U}} \leq_{\tilde{U}} A_I^2 \leq_{\tilde{U}} A_I^3 \leq_{\tilde{U}} A^4 \;\Rightarrow\; E(A_I^{1/2}) \geq E(A_I) \geq E(A_I^2) \geq E(A_I^3) \geq E(A_I^4).$$

The entropy expressions from E_1 to E_6, also including E_7 are considered to analyze the above IFS, from $A_I^{\frac{1}{2}}$ to A_I^4.

For each A-IFS the related A-GIFIx is obtained by the action of the arithmetic mean, taking into account the four A-GIFIx proposals Π_{LK}, Π_{RB}, Π_{GR} and Π_{KD}, whose expressions are described in Table 5. Thus, the presented entropy measures are compared based on these four expressions of the corresponding A-GIFIx, proposing the methodology related to the set of fuzzy implications $\{I_{LK}, I_{RB}, I_{GR}, I_{KD}\}$. The results related to fuzzy implication I_{LK} and the construction of the A-GIFIx $\Pi_{I_{LK}}$, considering the A-IFS from $A_I^{\frac{1}{2}}$ to $A_I 4$, are reported in Table 7.

Table 7 – A-IFE is Obtained from the A-GIFIx $\mathbf{\Pi_{LK}}$ with respect to N_S-Dual Construction

$\mathbf{\Pi_{LK}}$	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A_I^{\frac{1}{2}}$	0.3786	0.5016	0.5106	0.8660	0.3645	0.3686	0.0923
A_I^1	0.3810	0.4939	0.5054	0.8685	0.3564	0.3633	0.1200
A_I^2	0.3160	0.3953	0.4065	0.8437	0.3339	0.3407	0.1320
A_I^3	0.2700	0.3330	0.3438	0.8263	0.2512	0.2643	0.1424
A_I^4	0.2403	0.2938	0.3044	0.8147	0.2142	0.2313	0.1359

The last column in Table 7, indicated as E_7, summarizes the results for entropy obtained according to the following calculations:

$\Pi[A_I^{\frac{1}{2}}] = \{(6, 0.1311), (7, 0.1594), (8, 0.1198), (9, 0.0513), (10, 0.0)\}$; $E[A_I^{\frac{1}{2}}] = 0.09232$;

$\Pi[A_I^1] = \{(6, 0.1), (7, 0.2), (8, 0.2), (9, 0.1), (10, 0.0)\}$; $E[A_I^1] = 0.12$;

$\Pi[A_I^2] = \{(6, 0.03), (7, 0.16), (8, 0.28), (9, 0.19), (10, 0.0)\}$; $E[A_I^2] = 0.132$;

$\Pi[A_I^3] = \{(6, 0.007), (7, 0.098), (8, 0.336), (9, 0.271), (10, 0.0)\}$; $E[A_I^3] = 0.1424$;

$\Pi[A_I^4] = \{(6, 0.0015), (7, 0.0544), (8, 0.28), (9, 0.3439), (10, 0.0)\}$, $E[A_I^4] = 0.13596$.

Moreover, other methods based on the expression of A-GIFIx as $\mathbf{\Pi_{RB}}$, $\mathbf{\Pi_{GR}}$ and $\mathbf{\Pi_{KD}}$, obtained w.r.t. N_S-dual constructions can be analogously obtained.

By comparing the results achieved with the 07 methods, from E_1 to E_7, we can highlight that the entropy generated by Π_{LK} and Π_{RB} methods have the lowest entropy values for all A-IFS A_I^N. And, as a counterpart, the expression of entropy generated by Π_{GR} always generates the largest values for all A-IFS A_I^N.

4.4 Main Bibliographic References

Several research works have been studying entropy which attempts to quantify the uncertainty information modeled by A-IFS. A large amount of literature (Li; Chen; Huang, 2010; Wan, 2013; Li; Wan, 2014a,b) has been supporting the solutions for decision-making problems based on entropy measures analyses. So, the study of Atanassov's intuitionistic fuzzy entropy (A-IFE) introduced new formulas underlying the development of applications and interpretations on such research topics. Table 8 summarizes the main characteristics of related works considered in this thesis.

Table 8 – The Bibliographic Revision Integrating Entropy on A-IFS.

Paper/Year	Operators	Characterization
(Burillo; Bustince, 1996)	Aggregation Function	Providing a methodology to measure entropy for A-IFS
(Szmidt; Kacprzyk, 2001)	Aggregation Function	Defining an entropy based on the geometric interpretation for A-IFS
(Zhang; Jiang, 2008a)	Logarithmic Functions	Fuzzy entropy measuring only the derivation of membership and non-membership functions
(Bustince et al., 2011)	A-GIFIx	Structuring A-IFE based on the aggregation of generalized Atanassov's intuitionistic fuzzy index
(Wang; Wang, 2012)	Cotangent Function	Applying linguistic fuzzy multi-criteria decision-making method based on intuitionistic fuzzy entropy
(Xia; Xu, 2012)	Aggregation Function	Developing entropy and cross-entropy measures for intuitionistic fuzzy values
(Zhang, 2013)	Aggregation Function	Introducing axiomatic requirements for a set of A-IFE
(Pal et al., 2013)	Aggregation Function	Generating a new class of entropy measures on distinct facets of the uncertainty of A-IFS
(Mao; Yao; Wang, 2013)	Neperian Logarithm	Defining novel symmetric cross-entropy models to measure discrimination information
(Liu; Ren, 2014b)	Cosine Functions	Presenting an optimal model based on the minimum entropy principle
(Xiong et al., 2017)	Logarithmic Functions	Method based on the entropy weight approach contributing to determine MCDM weights
(Yuan; Zheng, 2022)	Deviation Operators	Improvements on A-IFE and application in the evaluation of regional collaborative innovation capability

(i) In 1996, Burillo and Bustince (Burillo; Bustince, 1996) defined intuitionistic fuzzy entropy to measure the degree of hesitation for intuitionistic fuzzy sets and on interval-valued fuzzy sets, providing a methodology to measure how far an A-IvIFS or A-IFS is from an FS.

(ii) In (Szmidt; Kacprzyk, 2001), a new non-probabilistic intuitionistic fuzzy entropy is concerned with the geometric interpretation for Atanassov's intuitionistic fuzzy sets.

(iii) In (Zhang; Jiang, 2008a), a measure of IF entropy generalizing of the De Luca and Termini proposal (Luca; Termini, 1972) is based on logarithmic fuzzy entropy, measuring only the derivation of membership and non-membership.

(iv) In (Bustince et al., 2011), the concept of Atanassov's intuitionistic fuzzy entropy is based on an aggregation of generalized Atanassov's intuitionistic fuzzy index (A-

GIFIx), which is characterized in terms of fuzzy implication operators to propose a construction method with order automorphisms.

(v) In (Xia; Xu, 2012), the authors developed entropy and cross-entropy measures for intuitionistic fuzzy values, discussing the properties of these measures and the relations between them and the existing ones. Aggregation operators are considered to treat the membership and non-membership information fairly, including practical examples to illustrate the developed methods.

(vi) In (Zhang, 2013), based on such distance measures and intuitionistic index, a set of entropy measures for IFS were proposed, verifying the axiomatic requirements given by Szmidt and Kacprzyk in 2001. The numerical examples demonstrated the efficiency of the proposed entropy methods for A-IFS.

(vii) Also in 2013, Pal et. al. (Pal et al., 2013) demonstrated that the existing measures of uncertainty for A-IFE cannot capture all facets of uncertainty associated with an A-IFS. Thus a generating family (class) of measures is proposed, where each family is illustrated with several examples.

(viii) In 2013, Mao et. al. discussed novel cross-entropy and symmetric cross-entropy models defined based on intuitionistic factors and fuzzy factor models to measure the discrimination of uncertain information. A constructive principle of entropy is refined and the relationship between cross-entropy and entropy is investigated. See also applications of pattern recognition and decision-making to demonstrate the efficiency of the cross-entropy and entropy models (Mao; Yao; Wang, 2013).

(ix) In (Liu; Ren, 2014b), a new intuitionistic fuzzy entropy modeling both the uncertainty and the hesitancy degree of A-IFS are based on cosine functions. The optimal model in sequence (Liu; Ren, 2015) is constructed according to the minimum entropy principle and practical examples are given to illustrate the effectiveness and practicability of the proposed method.

(x) In (Xiong et al., 2017), a generalized entropy measure for A-IFS based on logarithmic functions and A-IFIx. So, an efficient method based on the entropy weight approach is defined to determine the weights of decision makers (DM) and that attributes simultaneously. The proposed weight determination method can be applied to address the multi-attribute group decision-making (MADM) the problem in which the weight information is completely unknown.

(xi) In order to fully measure the fuzziness, the work reported in (Yuan; Zheng, 2022) considers the deviation between membership and non-membership and the influence of hesitation to construct the general expression of intuitionistic fuzzy entropy. Based on such expression, the regional collaborative innovation capability is evaluated, verifying the feasibility and practicability of the entropy.

Many other contributions can also be reported. See, e.g., in (Ye, 2010) presenting a proposal of two A-IFE using triangular functions. In (Verma; Sharma, 2013), an exponential A-IFE generalizes the exponential fuzzy entropy. And, in (Wei; Gao; Guo, 2012), an entropy measure using a cosine function is described.

4.5 Summary

This chapter studies the main reference describing the concepts of Atanassov's intuitionistic fuzzy sets underlying the notion of intuitionistic fuzzy entropy.

Main research on the study of entropy for A-IFS does not consider admissible linear orders to avoid the effort to generate a comparable group of sets to perform a comparison, as in the case described below. Moreover, the output result entropy in most of the proposal methods studied is reduced to a point value, losing the uncertainty and hesitation information, frequently presented on the input data of fuzzy systems.

In addition, this analysis shows relevant characteristics, discussed below:

(i) theoretical research using aggregation functions, similarity/dissimilarity measures also including distinct notions of distance measures;

(ii) applications on solving problems on pattern recognition, multi-criteria fuzzy decision making, classification tasks mainly connected to image processing; medical diagnosis and other technological areas;

(iii) Despite the proposals of new entropy formulas, mainly concerned with the amount of information and reliability in intuitionistic fuzzy sets, the presented applications do not make use of simulations in the methodology validation;

(iv) few research works develop studies integrating optimization and machine learning techniques.

The same structured study is an extension to other extensions of fuzzy logic, which here we consider A-IFL, IvFL, and A-IvIFL.

5 INTERVAL-VALUED FUZZY LOGIC

This chapter provides a brief account of Interval-valued Fuzzy Logic (IvFL). Firstly, partial and total orders are discussed on the set of all interval-valued fuzzy values $\mathbb{U}$. And in sequence, based on these relations, the definition of fuzzy connectives is presented regarding their main properties, which are relevant to the development of this theoretical study on admissible fuzzy connectives.

Basic concepts of interval-valued fuzzy negations and duality, including aggregation functions, fuzzy (co)implications, and conjugation operators related to automorphisms on $\mathbb{U}$ are also considered.

5.1 Basic Concepts of the Interval-valued Fuzzy Sets

Based on interpretations provided by the interval-valued fuzzy set theory, the membership degree of an element $x \in \chi$ to a fuzzy set corresponds to a value in the considered membership interval. So, we cannot say in a precise way what that value is, meaning that we just provide bounds for it represented by the interval-valued membership function.

Let $\mathbb{U} = \{[x_1, x_2] \colon x_1, x_2 \in U$ and $x_1 \leq x_2\}$ be the set of all subintervals of the unit interval $U = [0, 1]$. The projections $l_{\mathbb{U}}, r_{\mathbb{U}} : \mathbb{U} \to U$ are defined by

$$l_{\mathbb{U}}([x_1, x_2]) = x_1 \text{ and } r_{\mathbb{U}}, ([x_1, x_2]) = x_2, \ \forall x, y \in \mathbb{U} \tag{56}$$

and for $X \in \mathbb{U}$, $l_{\mathbb{U}}(X)$ and $r_{\mathbb{U}}(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively.

For each $x \in U$, the degenerate interval $[x, x]$ will be denoted by $\mathbf{x}$ and related set $\mathbb{D} = \{\mathbf{x} = [x, x] \colon x \in U\}$ denotes the set of all degenerate intervals on $\mathbb{U}$.

An interval-valued fuzzy set can be expressed as follows:

$$\mathbb{A} = \{(x, \mu_{\mathbb{A}}(x)) \colon x \in \chi \text{ and } \mu_{\mathbb{A}}(x) \in \mathbb{U}\}.$$

And, the set of all interval-valued fuzzy sets on the universe χ is denoted as $\mathcal{A}_{\mathbb{U}}$.

5.1.1 Partial Order Relations on $\langle \mathbb{U}, \leq_{\mathbb{U}} \rangle$

Among different order relations to compare elements in IvFSs (Gehrke; Walker; Walker, 1996), we take the component-wise **Kulisch-Miranker order** (or **product order**), given by:

$$X \leq_{\mathbb{U}} Y \Leftrightarrow \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y},\ \forall X, Y \in \mathbb{U}.$$

Thus, $\mathbf{0} \leq_{\mathbb{U}} X \leq_{\mathbb{U}} \mathbf{1}$, for all $X \in \mathbb{U}$. Moreover, for all $X, Y \in \mathbb{U}$, by taking

$$\wedge(X, Y) = \{\min(x, y) \colon x \in X, y \in Y\} \text{ and } \vee(X, Y) = \{\max(x, y) \colon x \in X, y \in Y\},$$

the structured set $\mathbb{U} \equiv (\mathbb{U}, \leq_{\mathbb{U}}, \wedge, \vee, \mathbf{1}, \mathbf{0})$ is a lattice.

We also consider the relation $\preccurlyeq_{\mathbb{U}} \subseteq \mathbb{U} \times \mathbb{U}$ given as

$$X \preccurlyeq_{\mathbb{U}} Y \Leftrightarrow \overline{X} \leq \underline{Y},\ \forall X, Y \in \mathbb{U}.$$

Therefore, we have that $X \preccurlyeq_{\mathbb{U}} Y \Rightarrow X \leq_{\mathbb{U}} Y$, $\forall X, Y \in \mathbb{U}$. And, both partial orders have $\mathbf{0} = [0, 0]$ and $\mathbf{1} = [1, 1]$ as the least and greatest elements, respectively.

Remark 5.1.1. *Since each interval $[\underline{X}, \overline{X}] \subseteq \mathbb{U}$ can be assigned uniquely to a point $(\underline{X}, \overline{X}) \in U \times U = U^2$, intervals can be ordered through pointwise orders in $U \times U$ induced by the partial order of intervals $\leq_{\mathbb{U}}$. Thus, when $K([0,1]) = \{(x, y) \in [0,1]^2 | x \leq y\}$, there is a natural bijection from $\mathbb{U}$ onto $K([0,1])$ resulting in the following*

$$[\underline{X}, \overline{X}] \leq_{\mathbb{U}} [\underline{Y}, \overline{Y}] \Leftrightarrow (\underline{X}, \overline{X}) \leq_{U \times U} (\underline{Y}, \overline{Y}),$$

and meaning that a partial (linear) order on $\mathbb{U}$ induces a partial (linear) order on the other, $K([0,1])$.

However, a linear order of intervals is required to compare anyone of its elements on $\mathbb{U}$. Thus, we consider an order relation extending the partial order $\leq_{\mathbb{U}}$ to a linear order by applying the notion of an admissible order.

5.1.2 Conjugation Operators on $\langle \mathbb{U}, \leq_{\mathbb{U}} \rangle$

In this session, some concepts of interval automorphisms are presented. This study is the basis for obtaining the conjugated functions, used in this work.

An interval function $\Phi_{\mathbb{U}} : \mathbb{U} \rightarrow \mathbb{U}$ is an **interval automorphism** (IvA) if it is bijective and monotonic with respect to the product order, that is, $X \leq_{\mathbb{U}} Y$ if and only if $X \leq_{\mathbb{U}} Y$.

Let $Aut(\mathbb{U})$ the set of all intervals in $\mathbb{U}$. Interval automorphisms are closed for composition, that is, $(\forall \Phi_{\mathbb{U}}, \Psi_{\mathbb{U}} \in Aut(\mathbb{U}),\ \Phi_{\mathbb{U}} \circ \Psi_{\mathbb{U}}) \in Aut(\mathbb{U})$; and $\forall \Phi_{\mathbb{U}} \in Aut(\mathbb{U})$, there is the reverse automorphism $\Phi_{\mathbb{U}}^{-1} \in \mathbb{U}$, such that $\Phi_{\mathbb{U}} \circ \Phi_{\mathbb{U}}^{-1} = Id_{\mathbb{U}}$. Thus, $(Aut(\mathbb{U}), \circ)$ is a group.

The action of an IvA $\Phi_{\mathbb{U}} : \mathbb{U} \rightarrow \mathbb{U}$ about an interval function $f_{\mathbb{U}} : \mathbb{U}^n \rightarrow \mathbb{U}$ is an interval function $f_{\mathbb{U}}^{\Phi_{\mathbb{U}}} : \mathbb{U} \to \mathbb{U}$, called **interval conjugated** of $f_{\mathbb{U}}$, defined by the expression:

$$f_{\mathbb{U}}^{\Phi_{\mathbb{U}}}(X_1, \dots, X_n) = \Phi_{\mathbb{U}}^{-1}(f_{\mathbb{U}}(\Phi_{\mathbb{U}}(X_1), \dots, \Phi_{\mathbb{U}}(X_n))). \quad (57)$$

5.1.3 Dual Operators on $(\mathbb{U}, \leq_{\mathbb{U}})$

Interval-valued fuzzy negations and dual operators are considered in the following.

Definition 5.1.1. *(Reiser et al., 2007) An interval function $\mathbb{N} : \mathbb{U} \to \mathbb{U}$ is an interval-valued fuzzy negation (IvFN) if, for all $X, Y \in \mathbb{U}$, it verifies the conditions:*

$\mathbb{N}1$**:** $\mathbb{N}([0,0]) = \mathbf{1}$*; e* $\mathbb{N}([1,1]) = \mathbf{0}$*;*

$\mathbb{N}2a$ *: If $X \geq Y$ then $\mathbb{N}(X) \leq \mathbb{N}(Y)$.*

$\mathbb{N}2b$ *: If $X \subseteq Y$ then $\mathbb{N}(X) \supseteq \mathbb{N}(Y)$.*

If $\mathbb{N}$ also satisfies the involutive property:

$\mathbb{N}3$ *: $\mathbb{N}(\mathbb{N}(X)) = X$, for all $X \in \mathbb{U}$,*

then $\mathbb{N}$ is called strong IvFN (Reiser et al., 2007).

Definition 5.1.2. *(Reiser; Bedregal; Reis, 2012) Let $\mathbb{N}$ an interval fuzzy strong negation in $\mathbb{U}$ and $f_{\mathbb{U}} : \mathbb{U}^n \leftrightarrow \mathbb{U}$ an interval function. The **interval function** $\mathbb{N}$**-dual** of $f_{N_{\mathbb{U}}}$ is given by:*

$$f_{N_{\mathbb{U}}}(X_1, \dots, X_n) = \mathbb{N}(f_{\mathbb{U}}(\mathbb{N}(X_1), \dots, \mathbb{N}(X_n))). \quad (58)$$

Example 5.1.1. *The interval extension of the standard negation $\mathbb{N}_S : \mathbb{U} \to \mathbb{U}$, w.r.t. to the Kulisch-Miranker's order, is given as:*

$$\mathbb{N}_S(X) = \mathbf{1} - X = [1 - \overline{X}, 1 - \underline{X}]. \quad (59)$$

The interval extension of fuzzy negation $N_2(x) = (1 - \sqrt{x})^2$ w.r.t. to the Kulisch-Miranker's order is, respectively, given as follows:

$$\mathbb{N}_2(X) = \left[\left(1 - \sqrt{\overline{X}}\right)^2, \left(1 - \sqrt{\underline{X}}\right)^2\right] \quad (60)$$

5.1.4 Aggregation Operators on $(\mathbb{U}, \leq_{\mathbb{U}})$

An interval-valued extension of an aggregation function $\mathbb{M} : \mathbb{U}^n \to \mathbb{U}$ demands the following conditions:

$\mathbb{M}1$**:** $\mathbb{M}(\mathbf{X}) = \mathbf{0} \Leftrightarrow \mathbf{X} = (\mathbf{0}, \dots, \mathbf{0})$; $\quad \mathbb{M}(\mathbf{X}) = \mathbf{1} \Leftrightarrow \mathbf{X} = (\mathbf{1}, \dots, \mathbf{1})$;

$\mathbb{M}2$**:** If $\mathbf{X} = (X_1, \dots, X_n) \leq_{\mathbb{U}^n} \mathbf{Y} = (Y_1, \dots, Y_n)$ then $\mathbb{M}(\mathbf{X}) \leq_{\mathbb{U}} \mathbb{M}(\mathbf{Y})$;

$\mathbb{M}3$**:** $\mathbb{M}(\mathbf{X}_\sigma) = \mathbb{M}(X_{\sigma_1}, \ldots, X_{\sigma_n}) = \mathbb{M}(X_1, \ldots, X_n) = \mathbb{M}(\mathbf{X})$, where σ is the permutation of the elements;

Interval-valued aggregations (IvA) are idempotent if they also verify the follows:

$\mathbb{M}4$: $\mathbb{M}(X, X) = X$, $\forall X \in \mathbb{U}$ (idempotency property).

Example 5.1.2. *Let* $\mathbb{M} : \mathbb{U}^n \to \mathbb{U}$ *be an idempotent IvA, together with the functions* $\bigwedge, \bigvee : \mathbb{U}^2 \to \mathbb{U}$*, given by:*

$$\bigwedge(X,Y) = [\wedge(\underline{X},\underline{Y}), \wedge(\overline{X},\overline{Y})] \quad \textit{and} \quad \bigvee(X,Y) = [\vee(\underline{X},\underline{Y}), \vee(\overline{X},\overline{Y})]. \tag{61}$$

The following inequation is held:

$$\bigwedge(X,Y) \leq \mathbb{M}(X,Y) \leq \bigvee(X,Y),\ \forall X,Y \in \mathbb{U}. \tag{62}$$

Let $\mathbb{M} : \mathbb{U}^2 \to \mathbb{U}$ be a binary IvA. By(Bedregal et al., 2017, Def. 3), its left and right projections are functions $\underline{\mathbb{M}}, \overline{\mathbb{M}} : U^2 \to U$ given as

$$\underline{\mathbb{M}}(x_1, x_2) = \underline{\mathbb{M}(\mathbf{x}_1, \mathbf{x}_2)} \text{ and } \overline{\mathbb{M}}(x_1, x_2) = \overline{\mathbb{M}(\mathbf{x}_1, \mathbf{x}_2)} \tag{63}$$

Example 5.1.3. *The interval expression of the quadratic-mean operator* $\mathbb{M}^* : \mathbb{U}^2 \to \mathbb{U}$ *given by Eq.(63) as follows*

$$\mathbb{M}^*(X,Y) = \left[\frac{1}{4}\left(\sqrt{\underline{X}} + \sqrt{\underline{Y}}\right)^2, \frac{1}{4}\left(\sqrt{\overline{X}} + \sqrt{\overline{Y}}\right)^2\right] \tag{64}$$

is an IvA verifying properties from $\mathbb{M}1$ *to* $\mathbb{M}4$*.*

In the following, the definition of an interval extension of conjuntive and disjuntive connectives on $\mathbb{U}$ is considered.

Definition 5.1.3. *(Bedregal et al., 2007) A function* $\mathbb{T}(\mathbb{S}) : \mathbb{U}^2 \to \mathbb{U}$ *is an IvT (IvS) if, for all* $X, Y, Z \in \mathbb{U}$ *the following properties are verified:*

$\mathbb{T}1$*:* $\mathbb{T}(X,Y) = \mathbb{T}(Y,X)$*;*	$\mathbb{S}1$*:* $\mathbb{S}(X,Y) = \mathbb{S}(Y,X)$*;*
$\mathbb{T}2$*:* $\mathbb{T}(X(\mathbb{T}(Y,Z)) = \mathbb{T}(\mathbb{T}(X,Y),Z)$*;*	$\mathbb{S}2$*:* $\mathbb{S}(X(\mathbb{S}(Y,Z)) = \mathbb{S}(\mathbb{S}(X,Y),Z)$*;*
$\mathbb{T}3$*:* $\mathbb{T}(X,1) = X$*;*	$\mathbb{S}3$*:* $\mathbb{S}(X,0) = X$*;*
$\mathbb{T}4$*:* $\mathbb{T}(X,Y) \leq \mathbb{T}(X,Z)$ *if* $Y \leq Z$	$\mathbb{S}4$*:* $\mathbb{S}(X,Y) \leq (X,Z)$ *if* $Y \leq Z$*.*

Proposition 5.1.1. *(Bedregal; Takahashi, 2006a) A function* $\mathbb{T}(\mathbb{S}) : \mathbb{U}^2 \to \mathbb{U}$ *is an IvT (IvS) if there are* T_1*,* $T_2(S_1$*,* $S_2) : U^2 \to U$ *such that* $T_1(x,y) \leq T_2(x,y)$ *(*$S_1(x,y) \leq S_2(x,y)$*) and the following holds:*

$$\mathbb{T}(X,Y) = [T_1(\underline{X},\underline{Y}), T_2(\overline{X},\overline{Y})] \qquad \mathbb{S}(X,Y) = [S_1(\underline{X},\underline{Y}), S_2(\overline{X},\overline{Y})] \tag{65}$$

In Proposition 5.1.1, an interval-valued t-(co)norm can be considered as an interval representation of a t-(co)norm. This generalization fits with the fuzzy principle, meaning that the interval-valued membership degree can be thought of as an approximation of the degree of exact relevance related to a specialist.

Thus, an IvT $\mathbb{T}$ is t-representable by t-norms $\mathbb{T}1$ and $\mathbb{T}2$, in the sense as proposed in (Deschrijver; Kerre, 2005; Cornelis; Deschrijver; Kerre, 2004). It is analogously stated in the dual construction of an IvS, as can be seen, in (Bedregal; Takahashi, 2006b).

5.1.5 Interval-valued Fuzzy (Co)Implications on $(\mathbb{U}, \leq_{\mathbb{U}})$

Fuzzy (co)implications can then be naturally extended to an interval-based approach. In the following, we study the definition and the main properties of interval-valued fuzzy (co)implication, shortly as IvI (IvC).

Definition 5.1.4. *(Bedregal et al., 2007) A function $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \to \mathbb{U}$ is an interval-valued fuzzy (co)implication if it satisfies the following conditions:*

$\mathbb{I}1$*: If $X \leq Z$ then $\mathbb{I}(X,Y) \geq \mathbb{I}(Z,Y)$;* $\mathbb{J}1$*: If $X \leq Z$ then $\mathbb{J}(X,Y) \geq \mathbb{J}(Z,Y)$;*

$\mathbb{I}2$*: If $Y \leq Z$ then $\mathbb{I}(X,Y) \leq \mathbb{I}(X,Z)$;* $\mathbb{J}2$*: If $Y \leq Z$ then $\mathbb{J}(X,Y) \leq \mathbb{J}(X,Z)$;*

$\mathbb{I}3$*: $\mathbb{I}(\mathbf{0},Y) = \mathbf{1}$;* $\mathbb{J}3$*: $\mathbb{J}(\mathbf{1},Y) = \mathbf{0}$;*

$\mathbb{I}4$*: $\mathbb{I}(X,\mathbf{1}) = \mathbf{1}$;* $\mathbb{J}4$*: $\mathbb{J}(X,\mathbf{0}) = \mathbf{0}$;*

$\mathbb{I}5$*: $\mathbb{I}(\mathbf{1},\mathbf{0}) = \mathbf{0}$;* $\mathbb{J}5$*: $\mathbb{J}(\mathbf{0},\mathbf{1}) = \mathbf{1}$.*

Since real numbers may be identified with degenerate intervals in the context of interval mathematics, the boundary conditions that must be satisfied by the classical fuzzy implications can be naturally extended to interval fuzzy degrees, whenever degenerate intervals are considered. So, an interval-valued fuzzy (co)implicator $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \to \mathbb{U}$ verifies $\mathbb{I}5$ ($\mathbb{J}5$) together with the following boundary conditions:

$\mathbb{I}0$: $\mathbb{I}(\mathbf{1},\mathbf{1}) = \mathbb{I}(\mathbf{0},\mathbf{0}) = \mathbb{I}(\mathbf{0},\mathbf{1}) = \mathbf{1}$; $\mathbb{J}0$: $\mathbb{J}(\mathbf{1},\mathbf{1}) = \mathbb{J}(\mathbf{1},\mathbf{0}) = \mathbb{J}(\mathbf{0},\mathbf{0}) = \mathbf{0}$;

Several reasonable properties may be required for fuzzy (co)implications. In this work, we consider the following ones:

$\mathbb{I}6$: $\mathbb{I}(\mathbf{1},Y) = Y$; $\mathbb{J}6$: $\mathbb{J}(\mathbf{0},Y) = Y$.

$\mathbb{I}7$: $\mathbb{I}(X,\mathbb{I}(Y,Z))=\mathbb{I}(Y,I(X,Z))$; $\mathbb{J}7$: $\mathbb{J}(X,\mathbb{J}(Y,Z))=\mathbb{J}(Y,J(X,Z))$;

$\mathbb{I}8$: $\mathbb{I}(X,Y)=\mathbf{1} \Leftrightarrow X \leq_{\mathbb{U}} Y$; $\mathbb{J}8$: $\mathbb{J}(X,Y)=\mathbf{0} \Leftrightarrow X \geq_{\mathbb{U}} Y$;

$\mathbb{I}9$: $\mathbb{I}(X,Y) = \mathbb{I}(\mathbb{N}(Y),\mathbb{N}(X))$, $\mathbb{N}$ is a SIFN; $\mathbb{J}9$: $\mathbb{J}(x,y) = \mathbb{J}(\mathbb{N}(Y),\mathbb{N}(X))$, $\mathbb{N}$ is a SIFN;

$\mathbb{I}10$: $\mathbb{I}(X,Y) = \mathbf{0} \Leftrightarrow X = \mathbf{1}$ and $Y = \mathbf{0}$; $\mathbb{J}10$: $\mathbb{J}(X,Y) = \mathbf{1} \Leftrightarrow X = \mathbf{0}$ and $Y = \mathbf{1}$.

The conditions under which an interval-valued fuzzy (co)implication can be obtained by a fuzzy (co)implication is studied in the proposition below:

Proposition 5.1.2. *(Baczyński; Jayaram, 2007, Prop 21) A fuzzy (co)implication $I(J) : U^2 \to U$ satisfies properties $I1$ ($J1$) and $I2$ ($J2$) if only if the interval fuzzy (co)implication $\mathbb{I}$ ($\mathbb{J}$) is given as*

$$\mathbb{I}(X,Y) = [I(\overline{X},\underline{Y}), I(\underline{X},\overline{Y})]; \qquad \mathbb{J}(X,Y) = [J(\overline{X},\underline{Y}), J(\underline{X},\overline{Y})]. \tag{66}$$

See in the Table **??**, the interval-valued extension of the fuzzy (co)implications presented in the Table 5. In addition, since the conditions of Proposition 5.1.2 are verified, these interval-valued fuzzy implications can be expressed by the corresponding fuzzy implications, as detailed in the following example.

Table 9 – Interval-valued Fuzzy Implications and $\mathbb{N}_S$-Dual Constructions.

Interval-valued Fuzzy Implications	Interval-valued Fuzzy Coimplications
$\mathbb{I}_{LK}(X,Y)=\begin{cases} \mathbf{1}, & \text{if } X \leq Y, \\ \mathbf{1} - X + Y, & \text{otherwise}; \end{cases}$	$\mathbb{J}_{LK}(X,Y)=\begin{cases} \mathbf{0}, & \text{if } X \geq Y, \\ Y - X, & \text{otherwise}; \end{cases}$
$\mathbb{I}_{KD}(X,Y)=\begin{cases} \mathbf{1}, & \text{if } X \leq Y, \\ \max(\mathbf{1} - X, Y), & \text{otherwise}; \end{cases}$	$\mathbb{J}_{KD}(X,Y)=\begin{cases} \mathbf{0}, & \text{if } X \geq Y, \\ \min(\mathbf{1} - X, Y), & \text{otherwise}; \end{cases}$
$\mathbb{I}_{RB}(X,Y)=\begin{cases} \mathbf{1}, & \text{if } X \leq Y, \\ \mathbf{1} - X + XY, & \text{otherwise}; \end{cases}$	$\mathbb{J}_{RB}(X,Y)=\begin{cases} \mathbf{0}, & \text{if } X \geq Y, \\ Y - XY, & \text{otherwise}; \end{cases}$
$\mathbb{I}_{GR}(X,Y)=\begin{cases} \mathbf{1}, & \text{if } X \leq Y, \\ \mathbf{0}, & \text{otherwise}; \end{cases}$	$\mathbb{J}_{GR}(X,Y)=\begin{cases} \mathbf{0}, & \text{if } X \geq Y, \\ \mathbf{1}, & \text{otherwise}; \end{cases}$

5.2 Main Bibliographic References

(i) In (Zhang; Zhang; Mei, 2009) a new axiomatic definition of entropy of IvFS is based on distance, and the relationship between entropy and similarity measure of IvFS is developed.

(ii) In (Bustince et al., 2019) a new class of similarity measures between interval-valued fuzzy sets with respect to total orders of intervals is studied; and, results consider the interval width, meaning that the uncertainty of the output is strongly related to the uncertainty of the input. For constructing the new interval-valued similarity, interval-valued aggregation functions and interval-valued re-

Table 10 – The Bibliographic Revision Integrating Entropy on IvFS.

Paper/Year	Operators	Characterization
(Zhang; Zhang; Mei, 2009)	Similarity Measures	Proposal of a new interval-valued entropy, investigating the distance notions and relationship on similarity measures of IvFS
(Bustince et al., 2019)	REF-Operator Aggregation	Defining entropy via similarity measures between interval-valued fuzzy sets w.r.t. total orders of intervals
(Bustince et al., 2019)	Aggregation Function	New definition of interval entropy considering the width of the membership intervals, based on normal functions
(Che; Suo; Li, 2021)	Distance measure	Entropy measure based on the relationship between distance functions and distance measures
(Takáč et al., 2019)	RDF-Operator Aggregation	The construction of distance and entropy measures are done by aggregating normal functions and applying for admissible orders
(Takáč et al., 2019)	Aggregation Function	Distance and entropy measures for IvFS are yielded w.r.t. total order as well
(Zhang et al., 2020)	Abstract Shadowed sets	The proposal of interval fuzzy entropy enables a new shadowed set model, namely, interval shadowed sets.
(Ohlan, 2022)	Exponential functions	Present a novel distance measures and the weighted exponential entropy measure

stricted equivalence functions (IvREF) which take into account the width of the intervals are discussed, and the results are applied to stereo image matching.

(iii) In (Bustince et al., 2019), a new definition of interval entropy takes into account the width of the membership intervals, by aggregating normal EN functions.

(iv) In (Che; Suo; Li, 2021), a new axiomatic definition of entropy measure from the graphical representation of the relationship between the distance function and the distance measure is explored, also illustrating an application in MCDM.

(v) In (Takáč et al., 2019), considering the width of intervals to connect the uncertainty of the output with the uncertainty of the input and making use of total orders between intervals, the construction of distance measures and entropy is done by aggregating interval-valued restricted dissimilarity functions (RDF) and interval-valued normal functions. An illustrative example in image thresholding uses the expression of the proposed interval entropy to show the validity of the proposal.

(vi) In (Takáč et al., 2019), both interval-valued restricted equivalence functions and interval-valued restricted dissimilarity functions are aggregated and similarity measures, distance and entropy measures for IvFS are yielded w.r.t. total order as well.

(vii) In (Zhang et al., 2020), a comprehensible method for measuring the interval fuzzy entropy is defined based on the notion of Shadowed sets. And also, the interval fuzzy entropy enables a new shadowed set model, namely, interval shadowed sets. By solving a fuzzy entropy loss minimization problem, a pair of optimal thresholds can be obtained.

(viii) In (Ohlan, 2022) the study of entropy and distance measures under an interval-valued intuitionistic fuzzy environment uses an exponential function. First, it presents the novel exponential entropy and distance measures for interval-valued intuitionistic fuzzy sets with proof of their authenticity. A method is offered to solve multi-criteria group decision-making (MCDM) problems in the IvFS environment based on the weighted exponential entropy measure.

5.3 Summary

The chapter reports the basic concepts of interval-valued fuzzy logic, which associates each element with a range of values instead of a single value, allowing for a more flexible and comprehensive representation of uncertainty and imprecision. This chapter also presents some relations as duality, conjugation, and representability based on the notion of partial orders on $\langle \mathbb{U}, \leq_{\mathbb{U}} \rangle$.

This study also considers the interval extension of aggregation, negation, and implication, including illustrations of the interrelation of such classes of operators and many examples. Concluding, the list of the main references supporting this revision is presented.

6 INTERVAL-VALUED INTUITIONISTIC FUZZY LOGIC

The Atanassov Interval-valued Intuitionistic Fuzzy Logic (A-IvIFL) is an extension of Intuitionistic Fuzzy Logic that takes into account the inherent uncertainty and imprecision associated with real-world data by employing interval-valued degrees on non-complementary relations of membership and non-membership functions. In A-IvIFL, instead of assigning precise real numbers as membership values, intervals are used to represent such degrees.

This flexible modeling of hesitant and uncertain information results in a powerful logical approach for decision-making processes, in environments where imprecise or incomplete data is prevalent.

This chapter delves into the key concepts and operators of Interval-valued Intuitionistic Fuzzy Logic, exploring its theoretical foundations and practical applications in areas such as decision support systems, expert systems, and pattern recognition.

6.1 Historical Approach

Atanassov and Gargov (Atanassov; Gargov, 1989) propose the interval-valued intuitionistic fuzzy logic (A-IvIFL) based on the notion of interval-valued intuitionistic fuzzy sets (A-IvIFS), addressing a mathematical and more intuitive method than classical logic, which can consider ambiguous or uncertainty, easily integrated with imprecise information.

A-IvIFL not only deals with the indecision inherent in natural language variables modeling computer systems but also collaborates with two other interpretations:

(i) The interpretation which is achieved when intervals may be considered as particular types of fuzzy sets, representing the imprecision of a variable depending on the computational context; and

(ii) The indecision about the relation between membership and non-membership degrees, not necessarily related as complementary degrees.

In this context, the former is concerned with calculations and numerical errors, regarding information related to various experts demanding membership and non-membership degrees. The latter is consistent with a non-zero intuitionistic fuzzy index.

Interval-valued Atanassov's intuitionistic fuzzy logic reinforces these interpretations by using the duality principle and aggregation operators, providing more flexible modeling for the truth associated with each variable. Thus, a more realistic analysis of the veracity of this variable can be inferred as much as from its interval membership degree and, from the complement, its relation with its interval non-membership degree.

Applications involve systems based on A-IvIFL together with computational tools, such as neural networks and evolutionary programming, expert systems, approximation reasoning, and digital image processing. Also noteworthy are applications in resource management, military strategies, medical diagnosis, pattern recognition, and clustering analysis in logistics (Bustince; Barrenechea; Mohedano, 2004).

However, despite the relevant advances, there is no consensus to consolidate a solution to guide the theoretical basis as well as the mathematical methods to support the area of decision-making based on multiple attributes.

This convergence is still a great research challenge, justified by many factors, among which the following stand out:

(i) insufficient knowledge of decision-makers;

(ii) the ever-increasing need to aggregate two or more possible judgments;

(iii) the challenging ability to deal with subjective characteristics of alternatives of fuzzy preference modeling supporting multi-attributes in decision making.

All these factors generate uncertain information that must be mapped from the modeling of decision-making systems based on multiple attributes (Dubois; Prade, 2000).

To compare data in this work, some results are focused on the study of generalized interval-valued Atanassov's intuitionistic fuzzy index (A-GIvIFIx) to obtain entropy measures, which are related to other parameters as distance, similarity, bissimilarity, correlation, accuracy, score, and many other ones performed over interval-valued Atanassov's intuitionistic fuzzy sets. Some authors put forward their axiomatic definitions in constructive methods to obtain interval-valued entropies for interval-valued Atanassov's intuitionistic fuzzy sets, including distance or similarity measures.

6.2 Main Concepts

Since Atanassov introduced the interval-valued fuzzy set theory, fruitful results have been achieved, introducing several basic operations, expanding both depth and scope and effectively aggregation and fuzzy connectives.

In the following, we review the definitions and basic results of the main interval-valued intuitionistic fuzzy connectives (and some generalizations), which are: fuzzy negations and duality relation, automorphisms, and conjugate operators. These connectives are important for the development and understanding of this work since the operators that define the entropy proposed in this thesis are constructed from these connectives.

The interval extension of generalized Atanassov's intuitionistic fuzzy index is studied, based on the notion of interval-valued fuzzy negation. The methodology to construct A-GIvIFIx by the composition of interval-valued fuzzy negations and interval-valued fuzzy (co)implications is presented, and some methods are presented by a selection of strong negations and selected classes of implications. In addition, some examples illustrating the constructive methods are presented.

6.2.1 Interval-valued Intuitionistic Fuzzy Sets

Based on (Atanassov; Gargov, 1998) and later in (Cornelis; Deschrijver; Kerre, 2004), we briefly report the main concepts and properties of interval-valued Atanassov's intuitionistic fuzzy sets (A-IvIFS shortly). An A-IvIFS $\mathbb{A}_I$ in a non-empty universe χ is expressed as:

$$\mathbb{A}_I = \{(\mathrm{x}, \nu_{\mathbb{A}_I}(\mathrm{x}), \nu_{\mathbb{A}_I}(\mathrm{x})) : \mathrm{x} \in \chi, \mu_{\mathbb{A}_I}(\mathrm{x}) + \nu_{\mathbb{A}_I}(\mathrm{x})) \leq \mathbf{1}\}, \tag{67}$$

and the set of all IvIFS is denoted by $\mathcal{A}_{\tilde{\mathbb{U}}}$. Thus, an intuitionistic fuzzy truth value of an element in $\mathbb{A}_I$ is related to the ordered pair $(\mu_{\mathbb{A}_I}(\mathrm{x}), \nu_{\mathbb{A}_I}(\mathrm{x}))$ where $\mathrm{x} \in \chi$.

When $\tilde{\mathbb{U}} = \{\widetilde{X} = (X_1, X_2) : (X_1, X_2) \in \mathbb{U}^2$ and $X_1 + X_2 \leq \mathbf{1}\}$[1] denotes the set of all Atanassov's interval-valued intuitionistic fuzzy degrees, the two order relations are considered:

$\tilde{X} \leq_{\tilde{\mathbb{U}}} \tilde{Y} \Leftrightarrow X_1 \leq Y_1$ and $X_2 \geq Y_2$;

$\tilde{X} \preccurlyeq_{\tilde{\mathbb{U}}} \tilde{Y} \Rightarrow X_1 \leq Y_1$ and $X_2 \leq Y_2$, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{U}}$;

we have that $(\tilde{\mathbb{U}}, \leq_{\tilde{\mathbb{U}}})$ and $(\tilde{\mathbb{U}}, \preccurlyeq_{\tilde{\mathbb{U}}})$ are partial ordered sets with $\tilde{\mathbf{0}} = (\mathbf{0}, \mathbf{1}) \leq_{\tilde{\mathbb{U}}} \tilde{X}$ and $\tilde{\mathbf{1}} = (\mathbf{1}, \mathbf{0}) \geq_{\tilde{\mathbb{U}}} \tilde{X}$ as the least and greatest elements on $\tilde{\mathbb{U}}$, respectivelly.

An Atanassov's interval-valued intuitionistic fuzzy degree has the projections $l_{\mathbb{I}_{\mathbb{I}}}, r_{\mathbb{I}_{\mathbb{I}}} : \tilde{\mathbb{U}} \to \mathbb{U}$ defined by

$$l_{\mathbb{I}_{\mathbb{I}}}(\tilde{X}) = X_1 \text{ and } r_{\mathbb{I}_{\mathbb{I}}}(\tilde{X}) = X_2.$$

When $X_1 + X_2 = \mathbf{1}$ then $\mathbb{A}_I$ is restricted to the set $\mathcal{A}_I$ of all interval-valued fuzzy sets.

A function $\tilde{\pi} : \chi \to \mathbb{U}$, called an interval-valued intuitionistic fuzzy index (A-IvIFIx) of an element $\mathrm{x} \in \chi$, related to an A-IvIFS A_I, is given as

$$\tilde{\pi}(\mathrm{x}) = \mathbb{N}_S(\mu_{\mathbb{A}_I}(\mathrm{x}) + \nu_{\mathbb{A}_I}(\mathrm{x})), \tag{68}$$

[1] We assume the component-wise addition on $\mathbb{U}$, see (Moore, 1979).

modeling not only the uncertainty degree but also the hesitancy (indeterminance) degree of $\mathrm{x} \in \chi$.

Thus, the accuracy function $\tilde{h} : \chi \rightarrow \mathbb{U}$ provides the interval-valued accuracy degree of $\mathrm{x} \in \chi$, given as $\tilde{h}(\mathrm{x}) + \tilde{\pi}(x) = \mathbf{1}$. So, it means that the larger $\tilde{\pi}(\tilde{h})$ the higher the hesitancy (accuracy) degree of $\tilde{\pi}(\mathrm{x})(\tilde{h}(\mathrm{x})) \in \mathbb{U}$.

Moreover, the difference between A_I and B_I is given by:

$$\mathbb{A}_I - \mathbb{B}_I = \{\tilde{X} = (\min(\nu_{\mathbb{A}_I}(\mathrm{x}), \nu_{\mathbb{B}_I}(\mathrm{x})), \max(\mu_{\mathbb{A}_I}(\mathrm{x}), \mu_{\mathbb{B}_I}(\mathrm{x}))) : \tilde{X} \in \tilde{\mathbb{U}}, x \in \chi\}.$$

6.2.2 Interval-valued Intuitionistic Conjugate Operator

A bijective and monotonic function $\Phi_{\tilde{\mathbb{U}}} : \tilde{\mathbb{U}} \rightarrow \tilde{\mathbb{U}}$ is an interval-valued intuitionistic automorphism on $\tilde{\mathbb{U}}$, meaning that below properties hold:

$\mathbb{A}_I 1$**:** $\Phi_{\tilde{\mathbb{U}}}(\tilde{\mathbf{1}}) = \tilde{\mathbf{1}}$ and $\Phi_{\tilde{\mathbb{U}}}(\tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$;

$\mathbb{A}_I 2$**:** $\Phi_{\tilde{\mathbb{U}}} \circ \Phi_{\tilde{\mathbb{U}}}^{-1}(\tilde{X}) = \tilde{X}$;

$\mathbb{A}_I 3$**:** $\tilde{X} \leq_{\tilde{\mathbb{U}}} \tilde{Y}$ if only if $\Phi_{\tilde{\mathbb{U}}}(\tilde{X}) \leq_{\tilde{\mathbb{U}}} \Phi_{\tilde{\mathbb{U}}}(\tilde{Y})$, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{U}}$.

In the set of all interval-valued intuitionistic automorphism ($Aut(\tilde{\mathbb{U}})$), the conjugate function of $f_{\tilde{\mathbb{U}}} : \tilde{\mathbb{U}}^n \rightarrow \tilde{\mathbb{U}}$ is a function $f_{\tilde{\mathbb{U}}}^{\Phi_{\tilde{\mathbb{U}}}} : \tilde{\mathbb{U}}^n \rightarrow \tilde{\mathbb{U}}$, defined as follows

$$f_{\tilde{\mathbb{U}}}^{\Phi_{\tilde{\mathbb{U}}}}(\tilde{\mathbf{X}}) = \Phi_{\tilde{\mathbb{U}}}^{-1}(f_{\tilde{\mathbb{U}}}(\Phi(\tilde{X}_1), \ldots, \Phi_{\tilde{\mathbb{U}}}(\tilde{X}_n))). \tag{69}$$

Reporting main results in (Costa; Bedregal; Neto, 2011, Theorem 17), let $\Phi_{\mathbb{U}} : \mathbb{U} \rightarrow \mathbb{U}$ be an interval-valued automorphism, $\Phi_{\mathbb{U}} \in Aut(\mathbb{U})$. Then, a $\Phi_{\mathbb{U}}$-representability of $\Phi_{\tilde{\mathbb{U}}}$ is given by

$$\Phi_{\tilde{\mathbb{U}}}(\tilde{X}) \quad = \quad (\Phi_{\mathbb{U}}(l_{\tilde{\mathbb{U}}}(\tilde{X})), \mathbf{1} - \Phi_{\mathbb{U}}(\mathbf{1} - r_{\tilde{\mathbb{U}}}(\tilde{X}))), \forall \tilde{X} \in \tilde{\mathbb{U}}; \tag{70}$$

Moreover, when $\Phi_{\mathbb{U}} \in Aut(\mathbb{U})$, for all $\tilde{X} \in \tilde{\mathbb{U}}$, a $\Phi_{\tilde{\mathbb{U}}}$-representability of $\Phi_{\tilde{\mathbb{U}}}$ is given by

$$\Phi_{\tilde{\mathbb{U}}}(\tilde{X}) \quad = \quad \left([\Phi_{\mathbb{U}}(\underline{X}_1), \Phi_{\mathbb{U}}(\overline{X}_1)], [1 - \Phi_{\mathbb{U}}(1 - \underline{X}_2, 1 - \Phi_{\mathbb{U}}(1 - \overline{X}_2]\right). \tag{71}$$

6.2.3 Interval-valued Intuitionistic Dual Conectives

An interval-valued intuitionistic fuzzy negation (IvIFN shortly) $\mathbb{N}_I : \tilde{\mathbb{U}} \rightarrow \tilde{\mathbb{U}}$ satisfies, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{U}}$, the following properties:

$\mathbb{N}_I$**1:** $\mathbb{N}_I(\tilde{\mathbf{0}}) = \mathbb{N}_I(\mathbf{0}, \mathbf{1}) = \tilde{\mathbf{1}}$ and $\mathbb{N}_I(\tilde{\mathbf{1}}) = \mathbb{N}_I(\mathbf{1}, \mathbf{0}) = \tilde{\mathbf{0}}$;

$\mathbb{N}_I$**2:** If $\tilde{X} \geq_{\tilde{\mathbb{U}}} \tilde{Y}$ then $\mathbb{N}_I(\tilde{x}) \leq_{\tilde{\mathbb{U}}} \mathbb{N}_I(\tilde{y})$.

Moreover, $\mathbb{N}_I$ is a strong IvIFN verifying the condition:

$\mathbb{N}_I3$: $\mathbb{N}_I(\mathbb{N}_I(\tilde{X})) = \tilde{X}, \forall \tilde{X} \in \tilde{\mathbb{U}}$.

Consider $\mathbb{N}_I$ as an IvIFN and $f_{\tilde{\mathbb{U}}} : \tilde{\mathbb{U}}^n \to \tilde{\mathbb{U}}$. The $\mathbb{N}_I$-dual interval-valued intuitionistic function of $f_{\tilde{\mathbb{U}}}$, denoted by $f_{\tilde{\mathbb{U}}} : \tilde{\mathbb{U}}^n \to \tilde{\mathbb{U}}$, is given by:

$$f_{N_{\tilde{\mathbb{U}}}}(\tilde{\mathbf{X}}) = \mathbb{N}_I(\Phi_{\tilde{\mathbb{U}}}(\mathbb{N}_I(\tilde{X}_1), \ldots, \mathbb{N}_I(\tilde{X}_n))), \forall \tilde{\mathbf{X}} = (\tilde{X}_1, \ldots, \tilde{X}_n) \in \tilde{\mathbb{U}}^n. \quad (72)$$

When $\tilde{\mathbb{N}}_I$ is a strong IvIFN, $\tilde{f}$ is a self-dual interval-valued intuitionistic function. And, by (Baczyński, 2004), taking a strong IvFN $\mathbb{N} : \mathbb{U} \to \mathbb{U}$, an IvIFN $\mathbb{N}_I : \tilde{\mathbb{U}} \to \tilde{\mathbb{U}}$ such that

$$\mathbb{N}_I(\tilde{X}) = (\mathbb{N}(\mathbb{N}_S(X_2)), \mathbb{N}_S(\mathbb{N}(X_1))), \quad (73)$$

is a strong IvIFN generated by means of the standard IvFN $\mathbb{N}_S$. Additionally, if $\mathbb{N} = \mathbb{N}_S$, Eq. (23) can be reduced to

$$\mathbb{N}_I(\tilde{X}) = (X_2, X_1) = [N_S(\overline{X}), N_S(\underline{X})].$$

Concluding this section, the complement of A-IvIFS $\mathbb{A}_I$ is defined by

$$\mathbb{A}_{I_C} = \{(\mathrm{x}, \mathbb{N}(\mathbb{N}_S(\nu_{\mathbb{A}_I}(\mathrm{x})), \mathbb{N}_S(N(\mu_{\mathbb{A}_I}(\mathrm{x})))) : \mathrm{x} \in \chi, \mu_{\mathbb{A}_I}(\mathrm{x}) + \nu_{\mathbb{A}_I}(\mathrm{x}) \leq 1\} \subseteq \mathcal{A}_{\tilde{U}} \quad (74)$$

6.3 Interval Extension of the A-GIFIx on $\langle \tilde{\mathbb{U}}, \leq_{\tilde{\mathbb{U}}} \rangle$

Since Atanassov's interval-valued intuitionistic fuzzy logic was introduced, many researchers have taken advantage of the interval-valued intuitionistic fuzzy index to represent not only the uncertainty but also the imprecision in modeling the membership and non-membership functions, which is strictly linked by interval-valued fuzzy connectives and relevant in the composition of the if-then rule of corresponding fuzzy system.

In intuitionistic fuzzy reasoning theory, intuitionistic fuzzy index operators play an important role. In this chapter, we introduce distinct expressions for A-IvIFIx operators which can be used in real applications, investigating properties, dual and conjugate constructions.

Focusing on the expressions of Atanassov's interval-valued intuitionistic fuzzy index based on the use of interval-valued fuzzy coimplications, a methodology to provide new expressions that preserve properties is considered.

Denoting a measure of non-determinacy, the intuitionistic fuzzy index of an element $x \in \chi$ in an interval-valued intuitionistic set $\mathbb{A}_I$, is conceived

In this section, we first introduced the axiomatic definition of a generalized interval-valued intuitionistic fuzzy index. In the sequence, its main properties and relationship with dual and conjugate operators are also discussed.

Definition 6.3.1. *A function $\tilde{\Pi} : \tilde{\mathbb{U}} \to \mathbb{U}$ is called a generalized Atanasso's interval-valued intuitionistic fuzzy index associated with a strong IvFN $\mathbb{N}$ A-GIvIFIx if, for all $X_1, X_2, Y_1, Y_2 \in \mathbb{U}$, it holds that:*

$\tilde{\Pi}1$: $\tilde{\Pi}(X_1, X_2) = \mathbf{1}$ *if only if* $X_1 = X_2 = \mathbf{0}$*;*

$\tilde{\Pi}2$: $\tilde{\Pi}(X_1, X_2) = \mathbf{0}$ *if only if* $X_1 + X_2 = \mathbf{1}$*;*

$\tilde{\Pi}3$: *If* $(Y_1, Y_2) \preccurlyeq_{\tilde{\mathbb{U}}} (X_1, X_2)$ *then* $\tilde{\Pi}(X_1, X_2) \leq_{\mathbb{U}} \tilde{\Pi}(Y_1, Y_2)$*;*

$\tilde{\Pi}4$: $\tilde{\Pi}(X_1, X_2) = \tilde{\Pi}(\mathbb{N}_I(X_1, X_2))$ *when* $\mathbb{N}$ *is a SIvFN.*

6.3.1 Obtaining A-GIvIFIx via Interval-valued Fuzzy Connectives

In (DA SILVA et al., 2018) is discussed the condition under which an interval-valued fuzzy (co)implication gives rise to generalized interval-valued Atanassov's intuitionistic fuzzy index associated with a strong IvFN, describing a new methodology to obtain different expressions of such operator by making use of fuzzy (co)implication operators and their dual constructors.

In the following, Theorem 6.3.1 extends main results in (Barrenechea et al., 2009).

Theorem 6.3.1. *(DA SILVA et al., 2018, Theorem 1) A function $\tilde{\Pi}_{\mathbb{N},\mathbb{I}}(\tilde{\Pi}_{\mathbb{N},\mathbb{J}}) : \tilde{\mathbb{U}} \to U$ is A-GIvIFIx iff exists a (co)implicator $\mathbb{I}(\mathbb{J}) : U^2 \to U$ verifying $\mathbb{I}1$($\mathbb{J}1$), $\mathbb{I}8$($\mathbb{J}8$), $\mathbb{I}9$($\mathbb{J}9$) and $\mathbb{I}10$($\mathbb{J}10$) such that*

$$\tilde{\Pi}_{\mathbb{I}}(X) = \mathbb{N}(\mathbb{I}(\mathbb{N}_{\mathbb{S}}(X_2), X_1)), \tag{75}$$

$$\tilde{\Pi}_{\mathbb{J}}(X) = \mathbb{J}(\mathbb{N}(\mathbb{N}_S(X_2)), \mathbb{N}(X_1)). \tag{76}$$

6.3.2 Dual Operators and A-IvIFIx with respect to IvFN

The Φ-representability and $\mathbb{N}$-dual A-GIvIFIx constructions are discussed in (DA SILVA et al., 2018) and reportes below.

Proposition 6.3.1. *(DA SILVA et al., 2018, Proposition 2) Let $\mathbb{I}_{\mathbb{N}}$ ($\mathbb{J}_{\mathbb{N}}$) $: U^2 \to U$ be the $\mathbb{N}$-dual operator of a (co)implication $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \to \mathbb{U}$. The following holds:*

$$\tilde{\Pi}_{\mathbb{I}_{\mathbb{N}}}(\tilde{\mathbb{X}}) = \tilde{\Pi}_{\mathbb{I}}(\tilde{\mathbb{X}}), \qquad \left(\tilde{\Pi}_{\mathbb{J}_{\mathbb{N}}}(\tilde{\mathbb{X}}) = \tilde{\Pi}_{\mathbb{J}}(\tilde{\mathbb{X}})\right). \tag{77}$$

In diagram of Figure 6 the following denotation is considered:
(i) $C(\mathbb{I})$ and $C(\mathbb{J})$denotes the class of all (co)implications;
(ii) $C(\mathbb{N})$ denotes the class of all negations;
(iii) $C(\tilde{\Pi})$ provides denotation to the class of all A-GIvIFIx;
In addition, the interrelations summarize the results stated in Theorem 6.3.1 and Proposition 6.3.1 are summarized in the diagram presented in Figure 6).

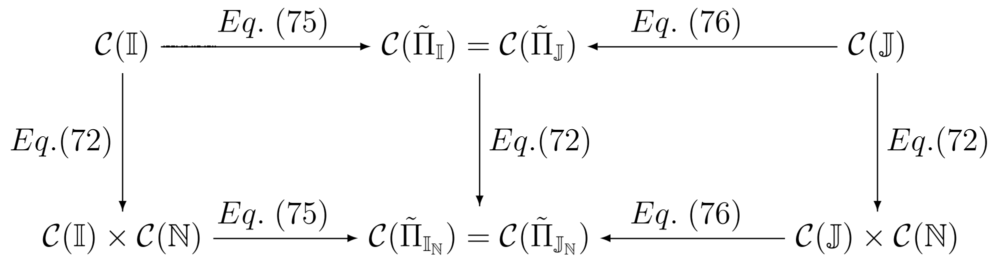

Figure 6 – Constructing A-GIvIFIx from Classes of Implications.

Corollary 6.3.1. *When $\mathbb{N} = \mathbb{N}_S$, Eq.(75) in Theorem 6.3.1 is given as*

$$\tilde{\Pi}_{\mathbb{I}}(\tilde{X}) = \mathbb{N}_S(\mathbb{I}(\mathbb{N}_S(X_2), X_1)) \tag{78}$$

$$\tilde{\Pi}_{\mathbb{J}}(\tilde{X}) = \mathbb{J}(X_2, \mathbb{N}_S(X_1) \tag{79}$$

Proposition 6.3.2. *Let $\mathbb{N}$ be an N-representable strong IvFN and $\Pi_{I(N,I)}\Pi_{I(J,N)} : \tilde{U} \to U$ be an A-IFIx(N,I). If $\mathbb{I}, \mathbb{J}$ are representable (co)implications given by Eq.(66), a function $\tilde{\Pi}_{\mathbb{I}} : \tilde{\mathbb{U}} \to \mathbb{U}$ given by Eq.(78) can be expressed as*

$$\tilde{\Pi}_{\mathbb{I}}(\tilde{X}) = [\Pi_{I(N,I)}(\overline{X}_2, \overline{X}_1), \Pi_{I(N,I)}(\underline{X}_2, \underline{X}_1)] \tag{80}$$

$$\tilde{\Pi}_{\mathbb{J}}(\tilde{X}) = [\Pi_{I(J,N)}(\overline{X}_2, \overline{X}_1), \Pi_{I(J,N)}(\underline{X}_2, \underline{X}_1)]. \tag{81}$$

Proof. We proof Eq.(80), the other one can be analogously done. By taking $X_1 = [\underline{X}_1, \overline{X}_1]$ and $X_2 = [\underline{X}_2, \overline{X}_2]$ then $X_1 + X_2 = [\underline{X}_1 + \underline{X}_2, \overline{X}_1 + \overline{X}_2] \leq [1,1]$, meaning that $\overline{X}_1 + \overline{X}_2 \leq 1$ and $\underline{X}_1 + \underline{X}_2 \leq 1$. Therefore $\Pi_{\mathbb{N},\mathbb{I}}(\tilde{X}) = \mathbb{N}(\mathbb{I}([1 - \overline{X}_2, 1 - \underline{X}_1], [\underline{X}_1, \overline{X}_1])) = [N(I(1 - \overline{X}_2, \overline{X}_1)), N(I(1 - \underline{X}_2, \underline{X}_1))]$. Concluding, $\tilde{\Pi}_{\mathbb{N},\mathbb{I}}(\tilde{X}) = [\Pi_{I(N,I)}(\overline{X}_2, \overline{X}_1), \Pi_{I(N,I)}(\underline{X}_2, \underline{X}_1)]$. So, Proposition 6.3.2 holds. □

Example 6.3.1. *Consider $\mathbb{I}_{RB}$ and related $\mathbb{N}_S$-dual construction $\tilde{\Pi}_{\mathbb{N}_S,\mathbb{J}_{RB}}$. By preserving the conditions of Proposition 6.3.2 and Eq.(76) we have that $\tilde{\Pi}_{\mathbb{N}_S,\mathbb{I}_{RB}}(X_1, X_2) = \tilde{\Pi}_{\mathbb{J}_{RB}}(X_1, X_2)$ and it can be expressed as*

$$\tilde{\Pi}_{\mathbb{N}_S,\mathbb{I}_{RB}}(X_1, X_2) = \begin{cases} 0, \text{ if } X_1 + X_2 = 1, \\ 1 - [1 - \overline{X}_2 - \overline{X}_1 + \overline{X}_2\overline{X}_1, 1 - \underline{X}_2 - \underline{X}_1 + \underline{X}_2\underline{X}_1], \textit{otherwise.} \end{cases} \tag{82}$$

Analogously, the methodology can be applied to other implications, obtaining other examples of generalized interval-valued intuitionistic fuzzy indexes associated with the interval extension of the standard negation.

Table 11, in the following, illustrates the method to obtain generalized interval-valued intuitionistic fuzzy indexes associated with the interval extension of the standard negation. Such examples consider the interval-valued fuzzy implications $\mathbb{I}_{KD}$, $\mathbb{I}_{LK}$, $\mathbb{I}_{RB}$ and $\mathbb{I}_{GR}$, presenting their algebraic expressions and corresponding A-GIvIFIx.

Table 11 – Generalized Interval-valued Intuitionistic Fuzzy Index Associated with the Standard Negation.

IvFI and $\mathbb{N}_S$-dual Constuctions	**A-GIvIFIx**
$\mathbb{I}_{KD}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \max(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$ $\mathbb{J}_{KD}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ \min(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$	$\tilde{\Pi}_{KD}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}-\max(X,Y), & \text{otherwise};\end{cases}$
$\mathbb{I}_{LK}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \mathbf{1}-X+Y, & \text{otherwise};\end{cases}$ $\mathbb{J}_{LK}(x,y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ Y-X, & \text{otherwise};\end{cases}$	$\tilde{\Pi}_{LK}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}-X-Y, & \text{otherwise};\end{cases}$
$\mathbb{I}_{RB}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \mathbf{1}-X+XY, & \text{otherwise};\end{cases}$ $\mathbb{J}_{RB}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ Y-XY, & \text{otherwise};\end{cases}$	$\tilde{\Pi}_{RB}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}-X-Y+XY, & \text{otherwise};\end{cases}$
$\mathbb{I}_{GR}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \mathbf{0}, & \text{otherwise};\end{cases}$ $\mathbb{J}_{GR}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ \mathbf{1}, & \text{otherwise};\end{cases}$	$\tilde{\Pi}_{GR}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}, & \text{otherwise};\end{cases}$

6.3.3 Relationship with Interval-valued Automorphisms

Proposition 6.3.3. *Let $\mathbb{N}^{\Phi_{\tilde{\mathbb{U}}}} : \mathbb{U} \to \mathbb{U}$ be the $\Phi_{\mathbb{U}}$-conjugate of a strong IvFN $\mathbb{N} : \mathbb{U} \to \mathbb{U}$ and $\Phi_{\mathbb{U}} : \mathbb{U} \to \mathbb{U}$ be a $\Phi_{\tilde{\mathbb{U}}}$-representable IvA given by Eq.(71). When $\Phi_{\tilde{\mathbb{U}}} : \tilde{\mathbb{U}} \to \tilde{\mathbb{U}}$ is a $\Phi_{\tilde{\mathbb{U}}}$-representable IvIFA given by Eq.(70), a function $\tilde{\Pi}^{\Phi_{\tilde{\mathbb{U}}}} : \tilde{\mathbb{U}} \to \mathbb{U}$ given by*

$$\tilde{\Pi}^{\Phi_{\tilde{\mathbb{U}}}}(X_1, X_2) \;=\; (\Phi^{-1}(\tilde{\Pi}(\Phi(X_1)), \mathbf{1} - \Phi(\mathbf{1} - X_2)), \tag{83}$$

is an A-GIvIFIx whenever $\tilde{\Pi} : \tilde{\mathbb{U}} \to \mathbb{U}$ is also an A-GIvIFIx.

Corollary 6.3.2. *Under the conditions of Proposition 6.3.3 and considering a $\Phi_{\mathbb{U}}$-representable IvA given by Eq.(71), we can express Eq.(83) as follows:*

$$\tilde{\Pi}^{\Phi_{\tilde{\mathbb{U}}}}(X_1, X_2) \;=\; \left[\tilde{\Pi}^{\Phi_{\mathbb{U}}}(\underline{X_1}, \underline{X_2}), \tilde{\Pi}^{\Phi_{\mathbb{U}}}(\overline{X_1}, \overline{X_2})\right]. \tag{84}$$

Proof. Straightforward Proposition 6.3.3. □

These results present the expression for the hesitation index based on interval-valued intuitionistic fuzzy implication and their dual construction.

Corollary 6.3.3. *Let $\Phi_{\tilde{\mathbb{U}}}$ be a $\Phi_{\tilde{\mathbb{U}}}$-representable automorphism in $Aut(\tilde{\mathbb{U}})$ and $\mathbb{I}^{\Phi_{\tilde{\mathbb{U}}}}(\mathbb{J}^{\Phi_{\tilde{\mathbb{U}}}}) : \tilde{\mathbb{U}}^2 \to \tilde{\mathbb{U}}$ be the corresponding $\Phi_{\mathbb{U}}$-conjugate operator related to a (co)implication $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \to \mathbb{U}$, verifying the conditions of Theorem 6.3.1. When $\mathbb{N}^{\Phi_{\mathbb{U}}}$ is a strong $\Phi_{\mathbb{U}}$-conjugate IvFN negation, the functions $\tilde{\Pi}_{\mathbb{I}}^{\Phi_{\tilde{\mathbb{U}}}}(\tilde{\Pi}_{\mathbb{J}}^{\Phi_{\tilde{\mathbb{U}}}}) : \tilde{\mathbb{U}} \to \mathbb{U}$ given by*

$$\tilde{\Pi}_{\mathbb{I}}^{\Phi_{\tilde{\mathbb{U}}}}(X_1, X_2) = \mathbb{N}^{\Phi_{\mathbb{U}}}(\mathbb{I}^{\Phi_{\tilde{\mathbb{U}}}}(\mathbb{N}_S(X_2), X_1)) \tag{85}$$

$$\tilde{\Pi}_{\mathbb{J}}^{\Phi_{\tilde{\mathbb{U}}}}(X_1, X_2) = \mathbb{J}^{\Phi_{\tilde{\mathbb{U}}}}(\mathbb{N}^{\Phi_{\mathbb{U}}}(\mathbb{N}_S(X_2), \mathbb{N}^{\Phi_{\mathbb{U}}}(X_1)) \tag{86}$$

are A-GIvIFIx.

Example 6.3.2. *Consider $\mathbb{I}_{RB}$ and related $\Phi_{\mathbb{U}}$-conjugate construction $\tilde{\Pi}_{\mathbb{N}_S,\mathbb{I}_{RB}}$ given by Eq.(82). For a $\Phi_{\tilde{\mathbb{U}}}$-representable automorphism given as $\Phi_{\tilde{\mathbb{U}}}(X) = X^n$, whenever $\mathbb{N}$ is a nonnegative integer, we have the following:*

$$\tilde{\Pi}_{\mathbb{N}_S,\mathbb{I}_{RC}}^{\Phi_{\tilde{\mathbb{U}}}}(X_1, X_2) \;=\; \left[\sqrt[n]{(1-\overline{X}_1^n)(1-\overline{X}_2)^n};\ \sqrt[n]{(1-\underline{X}_1^n)(1-\underline{X}_2)^n}\right]. \tag{87}$$

6.4 Relationship between A-GIFIx and A-GIvIFIx

The following proposition expresses the conditions under which an A-GIvIFIx $\tilde{\Pi} : \tilde{\mathbb{U}} \to \mathbb{U}$ can be representable by A-GIFIx $\Pi : \tilde{U} \to U$.

Theorem 6.4.1. *Let Π be a A-GIFIx. For $\tilde{X} = (X_1, X_2) \in \tilde{\mathbb{U}}$, the function $\tilde{\Pi} : \tilde{\mathbb{U}} \to \mathbb{U}$,*

$$\tilde{\Pi}(X_1, X_2) = [\Pi(\overline{X_1}, \overline{X_2}), \Pi(\underline{X_1}, \underline{X_2})], \tag{88}$$

is a $\tilde{\Pi}$-representable A-GIvIFIx.

Example 6.4.1. *Let $\Pi_{N_2} : \tilde{U} \to U$ be an A-GIFIx w.r.t. $N_2(x) = (1-\sqrt{x})^2$, which is given as*

$$\Pi_{N_2}(x_1, x_2) = \begin{cases} 0, \text{ if } x_1 + x_2 = 1; \\ \left(1 - \frac{1}{4}(\sqrt{x_2} - \sqrt{x_1})\right)^2, \text{ otherwise.} \end{cases}$$

So, by Theorem 6.4.1 and Eq.(60), we obtain the next $\tilde{\Pi}$-representable A-GIvIFIx :

$$\tilde{\Pi}_{\mathbb{N}_2}(X_1, X_2) = \begin{cases} \mathbf{0}, \text{ if } X_1 + X_2 = \mathbf{1}; \\ \left[\left(1-\frac{1}{4}(\sqrt{\overline{X}_2}-\sqrt{\overline{X}_1})\right)^2, \left(1-\frac{1}{4}(\sqrt{\underline{X}_2}-\sqrt{\underline{X}_1})\right)^2\right]. \end{cases} \tag{89}$$

Conversely, conditions under which an A-GIFIx can be obtained from an A-GIvIFIx are discussed in sequence.

Theorem 6.4.2. *Let* $\tilde{\Pi} : \tilde{\mathbb{U}} \to \mathbb{U}$ *be an A-GIvIFIx associated with an IvFN* $\mathbb{N}$*. Each function* $\underline{\tilde{\Pi}}, \overline{\tilde{\Pi}} : \tilde{U} \to U$ *given as*

$$\underline{\tilde{\Pi}}(x_1, x_2) = \underline{\tilde{\Pi}(\mathbf{x_1}, \mathbf{x_2})};\ \overline{\tilde{\Pi}}(x_1, x_2) = \overline{\tilde{\Pi}(\mathbf{x_1}, \mathbf{x_2})} \tag{90}$$

is an A-GIFIx, for all $\tilde{x} = (x_1, x_2) \in \tilde{U}$*.*

The actions of (Φ_1, Φ_2)-representable automorphisms on A-GIvIFIx provide a methodology enabling new expressions of A-GIvIFIx on $\tilde{\mathbb{U}}$ and preserving its main properties.

Proposition 6.4.1. *For* $\Phi_1, \Phi_2 \in Aut(\mathbb{U})$*,* $\Pi_{\Phi_1,\Phi_2} : \tilde{U} \to U$ *given as*

$$\Pi_{\Phi_1,\Phi_2}(\tilde{X}) = \Phi_1^{-1}(\Phi_2(1 - X_2) - \Phi_2(X_1)) \tag{91}$$

is an A-GIvIFIx w.r.t. strong IvIN $\mathbb{N}(X) = \Phi_2^{-1}(\mathbf{1} - \Phi_2(X))$*.*

Corollary 6.4.1. *Under the conditions of Proposition 6.4.1 the following holds:*

$$\tilde{\Pi}(\mathbf{0}, X_2) = \mathbf{1} - X_2 \Leftrightarrow \Phi_1(X) = \Phi_2(X), \forall \tilde{X} = (X_1, X_2) \in \tilde{\mathbb{U}}.$$

6.5 Main Bibliographic References

In the following, we highlight results obtained by a systematic revision of the literature on entropy measures and A-IvIFS, from 2015 to nowadays. The characteristics are pointed out in Table 12 and summarized in the following.

(i) In (Wei; Zhang, 2015), two proposals of entropy measures for A-IFS and IVFS based on the cosine function are introduced. They can overcome some shortcomings to measure both the fuzziness and intuitionism of these sets. As a result, the uncertain information can be described more sufficiently in applications to assess the experts' weights and to solve multi-criteria fuzzy group decision-making problems (MCDM).

(ii) In (Meng; Chen, 2015), an entropy measure is based on Shapley weighted similarity measures, exploring the interdependent or interactive characteristics between elements in IFS sets, defined by using the well-known Shapley functions, as an extension of the associated weighted similarity measures.

(iii) In (Xie; Lv, 2016), the definition and formula of entropy for A-A-IvIFS are proposed, including numerical examples verifying the appropriateness and effectiveness method for solving multi-attribute decision-making problems.

Table 12 – The Bibliographic Revision Integrating Entropy on IvFS.

Paper Reference	Operators	Characterization.
(Wei; Zhang, 2015)	Weight Average	Providing entropy measure based on distance and IFIx.
(Meng; Chen, 2015)	Shapley function	Defining an entropy-based on Shapley weighted similarity measures.
(Xie; Lv, 2016)	Aggregation	Entropy are studied considering fuzziness and lack of knowledge.
(Mao; Zhao; Ma, 2016)	Coef. Correlation	Compositive entropy combined fuzzy intuitionistic and span factors.
(Xian; Dong; Yin, 2017)	Weighted Averaging	Developing a new attribute weight based on the support and entropy measure of attribute values.
(Tiwari; Gupta, 2018)	Aggregation	Entropy and similarity concepts are based on probability and distance.
(Mishra; Chandel; Motwani, 2018)	Logarithmic	Entropy and divergence measures use the MABAC method.
(Wei et al., 2019)	Weight Average	Proposing a novel generalized exponential entropy measures.
(Tiwari, 2019)	Exponential	Defining the generalized similarity measures using a new entropy measure.
(Rani; Jain, 2019)	VIKOR operator	Study of a new entropy and divergence measures for A-IvIFS.

(iv) In (Mao; Zhao; Ma, 2016), combining intuitionistic, fuzzy, and span factors, the axiomatic definition of compositive entropy is proposed. Furthermore, such entropy is applied to MADM problems using the weighted correlation coefficient between A-IvIFS and pattern recognition by a similarity measure, which is transformed from the compositive entropy.

(v) In (Xian; Dong; Yin, 2017), a new attribute weight based on the support of entropy measure of attribute values is proposed, in order to determine the attribute weights in MCDM related to hesitant and imprecision information. Then, A-IvIFS are combined with weighted averaging (IvIFCWA) operators, whose interval-valued intuitionistic fuzzy numbers are concerned with the investment strategy, illustrating the validity and applicability of the proposed method.

(vi) In (Tiwari; Gupta, 2018), new axiomatic definitions of entropy measure applied concepts of probability and distance for A-IvIFS also consider the degree of hesitancy, which is consistent with the definition of entropy given by De Luca and Termini. The performance of proposed entropy and similarity measures on the basis of intuition are checked and compared with the existing entropy and similarity measures using numerical examples in the field of pattern recognition and medical diagnoses.

(vii) In (Mishra; Chandel; Motwani, 2018), a new integrated method based on the multi-attributive border approximation area comparison method is proposed (Mishra; Rani, 2017) using logarithmic functions. For the calculation of criteria weight, the subjective weights expressed by decision experts are aggregated and the proposed entropy and divergence measures method obtained more realistic weights. Considering a programming language selection problem, a sensitivity analysis with different weights of criteria showed the stability of the approach, which is efficient and consistent with the other methods.

(viii) In (Wei et al., 2019), a novel generalized exponential intuitionistic fuzzy entropy and generalized exponential interval-valued intuitionistic fuzzy entropy with interval area. The advantages of the new generalized entropy measures are compared with the existing A-IvIFE measures by some examples. The two novel generalized exponential entropy measures can distinguish the special cases well. The two novel generalized entropy measures are reasonable and more flexible than the existing entropy.

(ix) See in (Tiwari, 2019), the generalized entropy measure for A-IvIFS and relation are established to define the generalized similarity measures using the proposed entropy measure. Further, the proposed entropy measure is compared with some existing measures of entropy with the help of an illustrative example.

(x) In (Rani; Jain, 2019), the authors studied new entropy and divergence measures A-IvIFS and compared them with the existing measures. Further, to cope with the MCDM problems with non-commensurable and conflicting criteria, an extended VIKOR method is developed under an interval-valued intuitionistic environment. MCDM problem of supplier selection is discussed under incomplete and uncertain information situations, which employs its advantages and feasibility.

6.6 Summary

This chapter addressed the main concepts of interval-valued intuitionistic fuzzy logic and definitions of connectives (aggregation, negations, and implications) including their expression of interval-valued intuitionistic conjugate operators and dual constructions.

Next, the interval extension for the generalized intuitionistic fuzzy index (A-IvIFIx) was obtained as a composition of negations and aggregation operators. Moreover, we discuss the conditions under which an A-IvIFE was obtained by aggregation of an A-IvIFIx, measuring not only fuzziness but also the hesitation related to the complementary relation of interval-valued intuitionistic membership functions. In sequence, the relationship between A-GIFIx and A-GIvIFIx is presented.

At the end of this chapter, the bibliographic references underlying the study on A-IvIFL are presented. However, the above concepts were studied considering the notion of partial order on $\langle\tilde{\mathbb{U}}, \leq_{\tilde{\mathbb{U}}}\rangle$.

In Part III, we extend such methodology to obtain connectives and entropy in A-IvIFS by applying the notion of admissible orders on $\langle\tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}}\rangle$.

Part II

THEORETICAL CONTRIBUTIONS

7 CONCEPTS OF THE BOUNDED LATTICE $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$

This chapter studies of the concept of admissible linear orders on $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$, inspired by contributions given by (Matzenauer et al., 2021; Santana et al., 2020), which have obtained significant results in many applications based on multi-valued logic context (Bustince et al., 2013; Zapata et al., 2017; Matzenauer et al., 2022). And, in sequence, a new admissible order is presented.

7.1 Admissible orders on $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$

A partial order $\leq$ may be extended to an admissible order on $\mathbb{U}$ meaning that it is linear and refines $\leq$.

A linear order over $\mathbb{U}$ is a binary relation that is transitive, antisymmetric, and total. Equivalently, a linear order is a partial order under which every pair of intervals in $\mathbb{U}$ is comparable.

Definition 7.1.1. *(Bustince et al., 2013) The order $\preceq$ is called an admissible order on $\mathbb{U}$ if the following holds:*

(i) *$\preceq$ is a linear order on $\mathbb{U}$,*

(ii) *for all $X_1, X_2 \in \mathbb{U}$, $X_1 \preceq X_2$ whenever $X_1 \leq X_2$.*

The degenerate intervals $\mathbf{0}$ and $\mathbf{1}$ are the greatest and the smallest elements of $(\mathbb{U}, \preceq)$, respectively (Bustince et al., 2013).

7.1.1 Admissible orders obtained from aggregation functions

Based on Proposition 2.4 (Zapata et al., 2017), let $M_1, M_2 : \mathbb{U} \to \mathbb{U}$ be two aggregation functions such that $\forall X, Y \in \mathbb{U}$, the expressions $M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y})$ and $M_2(\underline{X}, \overline{X}) = M_2(\underline{Y}, \overline{Y})$ can only hold simultaneously if $X = Y$. The admissible order $\preceq_{M_1,M_2}$ on $\mathbb{U}$ is given by

$$X \preceq_{M_1 M_2} Y \Leftrightarrow \begin{cases} M_1(\underline{X}, \overline{X}) \leq M_1(\underline{Y}, \overline{Y}) \text{ or} \\ M_1(\underline{X}, \overline{X}) = M_1(\underline{Y}, \overline{Y}) \text{ and } M_2(\underline{X}, \overline{X}) \leq M_2(\underline{Y}, \overline{Y}). \end{cases} \quad (92)$$

The admissible orders reported in (Takáč et al., 2019) are described based on $\preceq_{M_1M_2}$-order.

Example 7.1.1. *Both relations* $\preceq_{Lex1}, \preceq_{Lex2} \in \mathbb{U}^2$*, respectively, given by:*

(i) $X \preceq_{Lex1} Y \Leftrightarrow \underline{X} < \underline{Y} \vee (\underline{X} = \underline{Y} \wedge \overline{X} \leq \overline{Y})$*;*

(ii) $X \preceq_{Lex2} Y \Leftrightarrow \overline{X} < \overline{Y} \vee (\overline{X} = \overline{Y} \wedge \underline{X} \leq \underline{Y})$*,* $\forall X, Y \in \mathbb{U}$*.*

are admissible orders refined by the lexicographical order of points in $\mathbb{R}^2$*. In this case,* $M_1(X) = \underline{X}$ *and* $M_2(X) = \overline{X}$*, meaning the left and right-projections.*

Example 7.1.2. *The linear order* $\preceq_{XY}$ *on* $\mathbb{U}$ *introduced by Xu and Yager in (Xu; Yager, 2006b) is an admissible order refining the Kulisch-Miranker's order* $\leq$*. For* $X, Y \in \mathbb{U}$*,*

$$X \preceq_{XY} Y \Leftrightarrow \begin{cases} \underline{X} + \overline{X} \leq \underline{Y} + \overline{Y} \text{ or} \\ \left(\underline{X} + \overline{X} = \underline{Y} + \overline{Y} \text{ and } \overline{X} - \underline{X} \leq \overline{Y} - \underline{Y}\right). \end{cases} \tag{93}$$

In this case, M_1 *and* M_2 *are the sum and difference functions, respectively.*

Definition 7.1.2. *(Bustince; Barrenechea; Pagola, 2008, Def. 3) For* $\alpha \in [0, 1]$*, a function* $K_\alpha : \mathbb{U} \to [0, 1]$ *is a* $K-$*operator if:*

K1 $K_\alpha(\mathbf{x}) = x$*, for all* $x \in [0, 1]$*;*

K2 $K_0(X) = \underline{X}$*,* $K_1(X) = \overline{X}$*, for all* $X \in \mathbb{U}$*;*

K3 *If* $X \leq Y$ *then* $K_\alpha(X) \leq K_\alpha(Y)$*, for all* $X, Y \in \mathbb{U}$ *and* $\alpha \in [0, 1]$*;*

K4 $\alpha \leq \beta$ *iff* $K_\alpha(X) \leq K_\beta(X)$*, for all* $X \in \mathbb{U}$*.*

Proposition 7.1.1. *Let* $K_\alpha : \mathbb{U} \to U$ *be a* $K-$*operator. When* W_X *denotes the amplitude of the interval* X*, for all* $X \in \mathbb{U}$*, a* $K-$*operator can be expressed as:*

$$K_\alpha(X) = K_0(X) + \alpha W(X). \tag{94}$$

Base on Eq.(94), K_α is a weighted mean, since $K_\alpha(X) = (1 - \alpha)\underline{X} + \alpha \overline{X}$.

Example 7.1.3. *When* $\alpha, \beta \in [0, 1]$ *with* $\alpha \neq \beta$*, based on the aggregation function* $K_\alpha(x, y) = (1 - \alpha)x + \alpha y$*. we can obtain the* $\preceq_{\alpha,\beta}$ *admissible order, refining to Kulisch-Miranker's order, just taking* $M_1(x, y) = K_\alpha(x, y)$ *and* $M_2(x, y) = K_\beta(x, y)$*.*

7.1.2 Fuzzy Conectives on $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$

A. Fuzzy Negations on $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$

Proposition 7.1.2. *(Zapata et al., 2017) By denoting $c = \frac{\underline{X}+\overline{X}}{2}$, $\alpha = \wedge(c, 1-c)$ and $r = \frac{\overline{X}-\underline{X}}{2}$, the function $\mathbb{N} : \mathbb{U} \to \mathbb{U}$ given as follows:*

$$\mathbb{N}(X) = [(1-c) - (\alpha - r), (1-c) + (\alpha - r)] \tag{95}$$

is a strong IvFN w.r.t. Xu-Yager's order given in Eq.(93).

Corollary 7.1.1. *The strong IvFN $\mathbb{N}$ w.r.t. Xu-Yager's order, given by Eq.(95), can also be expressed as follows*

$$\mathbb{N}_{XY}(X) = \begin{cases} \left[1 - \frac{\overline{X}+3\underline{X}}{2}, 1 - \frac{\overline{X}-\underline{X}}{2}\right], & \text{if } \overline{X} + \underline{X} \leq 1\,; \\ \left[\frac{\overline{X}-\underline{X}}{2}, 2 - \frac{3\overline{X}+\underline{X}}{2}\right], & \text{otherwise.} \end{cases} \tag{96}$$

B. Aggregation Functions on $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$

The aggregation functions w.r.t. Xu and Yager's admissible order are reported now to introduce a new class of interval-valued fuzzy implications, and their properties as discussed in the sequence.

Proposition 7.1.3. *(Zapata et al., 2017, Cor. 6.5) Let $\beta \in [0,1]$. The function M_β : $\mathbb{U}^n \to \mathbb{U}$ given as follows*

$$\mathbb{M}_\alpha(X,Y) = \begin{cases} \mathbf{0}, & \text{if } X = \mathbf{0} \text{ or } Y = \mathbf{0}, \\ \left[\beta\underline{X} + (1-\beta)\underline{Y}, \beta\overline{X} + (1-\beta)\overline{Y}\right], & \text{otherwise} \end{cases} \tag{97}$$

is an IvA w.r.t. Xu-Yager's order verifying $\mathbb{M}4$.

Corollary 7.1.2. *The function $\mathbb{M}_{\frac{1}{2}} : \mathbb{U}^n \to \mathbb{U}$ given as follows*

$$\mathbb{M}_{\frac{1}{2}}(X,Y) = \begin{cases} \mathbf{0}, & \text{if } X = \mathbf{0} \text{ or } Y = \mathbf{0}, \\ \left[\frac{1}{2}(\underline{X} + \underline{Y}), \frac{1}{2}(\overline{X} + \overline{Y})\right], & \text{otherwise;} \end{cases} \tag{98}$$

is also an aggregation w.r.t. Xu-Yager's order.

C. Fuzzy Implications on $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$

This subsection reports the notion of interval-valued fuzzy implication on $\langle \mathbb{U}, \preceq_{XY} \rangle$. Based on this admissible order, a new class of interval-valued fuzzy implications is presented, verifying properties that guarantee the construction of interval-valued intuitionistic fuzzy index in the next chapters.

Proposition 7.1.4. *(Zapata et al., 2017, Prop.5.8) Let* $\mathbb{M}:\mathbb{U}^2\to\mathbb{U}$ *be an IvA verifying* $\mathbb{M}4$ *and* $\mathbb{N}:\mathbb{U}\to\mathbb{U}$ *be an IvN w.r.t. Kulish-Miranker order. The function* $\mathbb{I}_{\mathbb{M},\mathbb{N}}:\mathbb{U}^2\to\mathbb{U}$ *given by the following expression:*

$$\mathbb{I}_{\mathbb{M},\mathbb{N}}(X,Y) \;=\; \begin{cases} \mathbf{1}, \textit{ if } X \preceq Y; \\ \mathbb{M}(\mathbb{N}(X),Y), \textit{ otherwise}. \end{cases} \tag{99}$$

is an IvFI w.r.t. an admissible $\preceq$*-order.*

Proposition 7.1.5. *Under the conditions of Proposition 7.1.4, when* $\mathbb{N}$ *is a strong IvFN, the IvFI* $\mathbb{I}_{\mathbb{M},\mathbb{N}}$ *in Eq.(99) satisfies the properties* $\mathbb{I}4$*,* $\mathbb{I}5$ *and* $\mathbb{I}6$ *w.r.t. (partial/total) order* $\preceq$*.*

Proof. For all $X,Y\in\mathbb{U}$ the following holds:
$\mathbb{I}4$: Straightforward.
$\mathbb{I}5$: For $X,Y\in\mathbb{U}$, if $\mathbb{N}(Y)\preceq\mathbb{N}(X)$ or $X\preceq Y$ we obtain that $\mathbb{I}_{\mathbb{M},\mathbb{N}}(\mathbb{N}(Y),\mathbb{N}(X)))=\mathbf{1}=\mathbb{M}(\mathbb{N}(X),Y)$. Otherwise, if $X\prec Y$, $\mathbb{I}_{\mathbb{M},\mathbb{N}}(\mathbb{N}(Y),\mathbb{N}(X)))=\mathbb{M}(\mathbb{N}(\mathbb{N}((Y)),\mathbb{N}(X))=\mathbb{M}(\mathbb{N}(X),Y)$. Thus, $\mathbb{I}_{\mathbb{M},\mathbb{N}}(\mathbb{N}(Y),\mathbb{N}(X))=\mathbb{I}_{\mathbb{M},\mathbb{N}}(X,Y)$.
$\mathbb{I}6$: By $\mathbb{I}1$b, it holds that $\mathbb{I}_{\mathbb{M},\mathbb{N}}(\mathbf{1},\mathbf{0})=\mathbf{0}$. For all $X,Y\in\mathbb{U}$, by $\mathbb{M}1,\mathbb{M}2$ and $\mathbb{N}2$, we have that: $\mathbb{I}_{\mathbb{M},\mathbb{N}}(X,Y))=\mathbf{0}\Rightarrow\mathbb{M}(\mathbb{N}(X),Y)=\mathbf{0}\Rightarrow X=\mathbf{1}\wedge Y=\mathbf{0}$. Thus, Prop. 7.1.5 is held. □

The new class of implicantion $\mathbb{I}(\mathbb{M}_\beta,\mathbb{N})$ w.r.t. $\preceq_{XY}$ is presented now.

Proposition 7.1.6. *The function* $\mathbb{I}_{\mathbb{M}_\beta,\mathbb{N}}:\mathbb{U}^2\to\mathbb{U}$ *given as follows:*

$$\mathbb{I}_{\mathbb{M}_\beta,\mathbb{N}}(X,Y)=\begin{cases} \mathbf{1},\ \textit{if } X \preceq_{XY} Y; \\ \mathbf{0},\ \textit{if } X=\mathbf{1} \textit{ and } Y=\mathbf{0}; \\ \left[\beta\left(1-\frac{\overline{X}+3\underline{\underline{X}}}{2}\right)+(1-\beta)\underline{Y},\beta\left(1-\frac{\overline{X}-\underline{\underline{X}}}{2}\right)+(1-\beta)\overline{Y}\right], \\ \quad \textit{if } \overline{X}+\underline{X}<1 \textit{ and } Y\prec_{XY} X; \\ \left[\beta\frac{\overline{X}-\underline{\underline{X}}}{2}+(1-\beta)\underline{Y},\beta\left(2-\frac{3\overline{X}+\underline{\underline{X}}}{2}\right)+(1-\beta)\overline{Y}\right], \textit{ otherwise}. \end{cases} \tag{100}$$

is IvFI w.r.t. Xu-Yager's admissible order given by Eq.(93).

Proof. It follows from results of Proposition 7.1.5 and Eq.(99) in Proposition 7.1.4, taking $\mathbb{M}_\beta$ and $\mathbb{N}$ in Eqs.(97) and (96), respectively. □

And now, a member in the class interval-valued fuzzy implications $\mathbb{I}_{\mathbb{M}_\beta,\mathbb{N}}$, generated by the fuzzy aggregation ($\mathbb{M}_\beta$ and fuzzy negation $\mathbb{N}_{XY}$) is obtained straightforward Eqs.(98) and (96).

Example 7.1.4. *The function* $\mathbb{I}_{\mathbb{M}_{\frac{1}{2}},\mathbb{N}_{XY}}:\mathbb{U}^2\rightarrow\mathbb{U}$ *given as follows:*

$$\mathbb{I}_{\mathbb{M}_{\frac{1}{2}},\mathbb{N}_{XY}}(X,Y) \;=\; \begin{cases} \mathbf{1}, \text{ if } X \preceq Y; \\ \mathbf{0}, \text{ if } X=\mathbf{1} \text{ or } Y=\mathbf{0}; \\ \frac{1}{2}\left[1+\underline{Y}-\frac{\overline{X}+3\underline{X}}{2}, 1+\overline{Y}-\frac{\overline{X}-\underline{X}}{2}\right], \text{ if } \overline{X}+\underline{X}<1 \text{ and } Y \prec X; \\ \frac{1}{2}\left[\underline{Y}-\frac{\overline{X}-\underline{X}}{2}, 1+\overline{Y}-\frac{3\overline{X}+\underline{X}}{2}\right], \text{ otherwise.} \end{cases} \tag{101}$$

is an IvFI w.r.t. Xu-Yager's order.

In particular, another construction for the class of interval-valued fuzzy implications can be described, when the partial order on $\langle\mathbb{U},\leq_{\mathbb{U}}\rangle$ is considered. See, an illustration in the following example.

Example 7.1.5. *Let* $\mathbb{M}^*$ *be IvA given in Eq.(64) and* $\mathbb{N}_2$ *be the strong IvFN given in Eq.(60). The function* $\mathbb{I}_{\mathbb{M}^*,\mathbb{N}_2}:\mathbb{U}^2\rightarrow\mathbb{U}$ *is an IvFI w.r.t. Kulisch-Miranker's order obtained by Eq.(99) in Proposition 7.1.4 and given as:*

$$\mathbb{I}_{\mathbb{M}^*,\mathbb{N}_2}(X_1,X_2) \;=\; \begin{cases} \mathbf{1}, \text{ if } X_1 \leq_{\mathbb{U}} X_2; \\ \left[\frac{1}{4}\Big(1-\sqrt{\underline{X}_2}-\sqrt{\overline{X}_1})\Big)^2, \frac{1}{4}\Big(1-\sqrt{\overline{X}_2}-\sqrt{\underline{X}_1}\Big)^2\right], \text{ otherwise.} \end{cases} \tag{102}$$

7.2 Admissible order on $\langle\mathbb{U},\preceq_A\rangle$

This section contributes with concepts on interval-valued fuzzy connectives w.r.t. a class of admissible order generated from injective functions. In particular, the detailed discussion on the admissible interleaving operators is also considered.

The next is a method for generating admissible orders based on an injective function A.

Theorem 7.2.1. *Let* $A\colon\mathbb{U}\rightarrow[0,1]$ *be a function and* $A(\mathbf{0})=0$ *and* $A(\mathbf{1})=1$*. The* $\preceq_A$*-relation on* $\mathbb{U}$ *given by*

$$X \preceq_A Y \Leftrightarrow \begin{cases} X=Y, \text{ or} \\ A(X)<A(Y), \end{cases} \tag{103}$$

is a bounded partial order on $\mathbb{U}$*. In addition,* $\preceq_A$ *is an admissible order if and only if* A *is injective and increasing w.r.t. the product and usual order on* $\mathbb{U}$ *and* $[0,1]$*, respectively.*

Proof. The relation $\preceq_A$ is reflexive and antisymmetric, immediately. Let $X,Y,Z\in\mathbb{U}$ such that $X\preceq_A Y$ and $Y\preceq_A Z$. Then, the following hold:
(i) In case $X=Y$ or $Y=Z$, we immediately have that $X\preceq_A Z$.

(ii) If $X \neq Y$ and $Y \neq Z$ then $X \prec_A Y \prec_A Z$. So, $A(X) < A(Y) < A(Z)$ and consequently $A(X) < A(Z)$. Thereby, $X \prec_A Z$.

Based on the above cases, we have a transitive $\langle \mathbb{U}, \preceq_A \rangle$-relation. Hence, since, $A(\mathbf{0}){=}0{\leq}A(X){\leq}1 = A(\mathbf{1})$, then $\langle \mathbb{U}, \preceq_A \rangle$ is a bounded partial order.
In addition, if A is injective then, for each $X, Y \in \mathbb{U}$, we have three cases:
(i) In case $A(X) < A(Y)$, it implies that $X \preceq_A Y$;
(ii) In case $A(Y) < A(X)$, it implies $Y \preceq_A X$; and
(iii) In case $A(X) = A(Y)$ then, since A is an injective function, it results on $X = Y$. Therefore, if A is injective then $\preceq_A$ is a linear order.

Finally, consider $X, Y \in \mathbb{U}$, such that $X \leq Y$. If $X = Y$ then, trivially, $X \preceq_A Y$. If $X < Y$ then $A(X) < A(Y)$, since A is injective and increasing. Therefore, $X \preceq_A Y$ and so, the $\preceq_A$-order refines the usual $\leq$-order. Concluding, $\preceq_A$-order is an admissible order on $\mathbb{U}$ whenever A is injective and increasing w.r.t. product order. Conversely, if $\preceq_A$ is a linear order then for each $X, Y \in \mathbb{U}$ such that $X \neq Y$ then either $X \prec_A Y$ or $Y \prec_A X$. In both cases, from Eq.(103), we have that $A(X) \neq A(Y)$. Besides, if $\preceq_A$ is admissible then for each $X, Y \in \mathbb{U}$ such that $X \leq Y$ then $X \preceq_A Y$ and therefore $A(X) \leq A(Y)$, i.e. A is increasing w.r.t. product order. □

In Theorem 7.2.1, since $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$ then when A is injective, the following additional properties can be considered:

$(\mathbb{A}0)$ $A(X) = 0$ iff $X = \mathbf{0}$; $(\mathbb{A}1)$ $A(X) = 1$ iff $X = \mathbf{1}$.

Now, a method for generating admissible orders $\langle \mathbb{U}, \preceq_A \rangle$ by the injective functions named decimal-digit interleaving (DDI), is discussed below.

Firstly, observe that there is a convention to represent each real number as an infinite string of decimal digits, where decimals like 0.25 are represented by the infinite string $0.2499\ldots = 0.24\tilde{9}$. However, here we will represent 0.25 by $0.2500... = 0.25\tilde{0}$, but we will omit the $\tilde{0}$ in this context when it is convenient for the sake of simplicity. The only exception is 1.0 which will be represented by $0.99\ldots = 0.\tilde{9}$. Obviously, when the real number is irrational or an infinite rational, the infinite string of decimal digits representing its are the usual one. For example, for $\frac{1}{6}$ such string is $0.166.... = 0.1\tilde{6}$. And, the i-th decimal digit of this representation of a real number $x \in [0, 1]$ will be denoted by $x^{[i]}$.

See, the same representation can be given to sub-intervals of the unit interval $[0, 1]$. For that, consider the infinite decimal expansion, described above, of the endpoints of an interval $X = \left[\underline{X}, \overline{X}\right] \subseteq [0, 1]$, which is, indicated as:

$$\left[\underline{X}, \overline{X}\right] = \left[0.\underline{X}^{[1]}\underline{X}^{[2]} \ldots \underline{X}^{[n]}..., \, 0.\overline{X}^{[1]}\overline{X}^{[2]} \ldots \overline{X}^{[n]}...\right]. \tag{104}$$

Thus, $[0.25, 0.4]$ is represented by $[0.25\tilde{0}, 0.4\tilde{0}]$.

In the next definitions, we consider two orderings for interleaving the digits comprising the numbers $\underline{X}$ and $\overline{X}$, which are the corresponding extremes of the subinterval $X \subseteq [0,1]$. These interleaves are related to the same position in their decimal expansions.

Definition 7.2.1. *The DDI functions* $\overrightarrow{\mathbf{A}}, \overleftarrow{\mathbf{A}} : \mathbb{U} \to [0,1]$ *are, respectively, given by*

$$\overrightarrow{\mathbf{A}}(X) = \begin{cases} 0.\underline{X}^{[1]}9\underline{X}^{[2]}9\ldots, & \text{if } \overline{X} = 1; \\ 0.\underline{X}^{[1]}\overline{X}^{[1]}\underline{X}^{[2]}\overline{X}^{[2]}\ldots, & \text{otherwise; and} \end{cases} \tag{105}$$

$$\overleftarrow{\mathbf{A}}(X) = \begin{cases} 0.9\underline{X}^{[1]}9\underline{X}^{[2]}9\ldots, & \text{if } \overline{X} = 1; \\ 0.\overline{X}^{[1]}\underline{X}^{[1]}\overline{X}^{[2]}\underline{X}^{[2]}\ldots, & \text{otherwise.} \end{cases} \tag{106}$$

Example 7.2.1. *Take* $X = [0.3, 0.72]$, $Y = [0.127, 0.4]$ *and* $Z = [0.9, 1]$*, then :*
(i)$\overrightarrow{\mathbf{A}}(X) = 0.3702$ *and* $\overleftarrow{\mathbf{A}}(X) = 0.732$*;*
(ii) $\overrightarrow{\mathbf{A}}(Y) = 0.14207$ *and* $\overleftarrow{\mathbf{A}}(Y) = 0.410207$*; and*
(iii) $\overrightarrow{\mathbf{A}}(Z) = 0.90\widetilde{9}$ *and* $\overleftarrow{\mathbf{A}}(Z) = 0.99\widetilde{90}$.

Proposition 7.2.1. *The functions* $\overrightarrow{\mathbf{A}}, \overleftarrow{\mathbf{A}} : \mathbb{U} \to [0,1]$ *given in Eq. (105) and Eq.(106) are both injective functions, satisfying the boundary conditions* $\mathbb{A}0$ *and* $\mathbb{A}1$ *and are increasing w.r.t. the product* $\leq$*-order, i.e.*

$$X \leq Y \Rightarrow \overrightarrow{\mathbf{A}}(X) \leq \overrightarrow{\mathbf{A}}(Y);\; X \leq Y \Rightarrow \overleftarrow{\mathbf{A}}(X) \leq \overleftarrow{\mathbf{A}}(Y);\; \text{and } X \leq Y \Rightarrow \overrightarrow{\mathbf{A}}(X) \leq \overleftarrow{\mathbf{A}}(Y).$$

Proof. Analogous to (Santana et al., 2020, Proposition 1). □

From now, a function $A \colon \mathbb{U} \to [0,1]$ such that $A(\mathbf{0}) = 0$, $A(\mathbf{1}) = 1$, injective and increasing w.r.t. product and usual order on $\mathbb{U}$ and $[0,1]$ will be called just by admissible interleaving. Thereby, $\overrightarrow{\mathbf{A}}$ and $\overleftarrow{\mathbf{A}}$ are admissible interleaving. In the following, we will denote $\overrightarrow{\mathbf{A}}(X)$ by $\mathbf{A}(X)$.

Example 7.2.2. *For* $X = [0.\tilde{3}, 0.72]$, $\mathbf{A}(X) = 0.3732\widetilde{30}$ *and* $\overleftarrow{\mathbf{A}}(X) = 0.7320\widetilde{03}$.

Corollary 7.2.1. *The relations* $\preceq_{\overleftarrow{\mathbf{A}}}$ *-order and* $\preceq_{\mathbf{A}}$*-order given as*

$$X \preceq_{\mathbf{A}} Y \Leftrightarrow \mathbf{A}(X) \leqslant \mathbf{A}(Y) \text{ and } X \preceq_{\overleftarrow{\mathbf{A}}} Y \Leftrightarrow \overleftarrow{\mathbf{A}}(X) \leqslant \overleftarrow{\mathbf{A}}(Y) \tag{107}$$

are admissible orders on $\mathbb{U}$.

Proof. Straight from Theorem 7.2.1 and Proposition 7.2.1. □

Example 7.2.3. *Let* $X = [0.15, 0.88], Y = [0.16, 0.86] \in \mathbb{U}$*. We have that:*
(i) $X \preceq_{Lex1} Y$, $Y \preceq_{Lex2} X$ *and* $Y \preceq_{XY} X$*; and*

(ii) Since $\mathbf{A}(X) = 0.1858 < 0.1866 = \mathbf{A}(Y)$*, then* $X \preceq_{\mathbf{A}} Y$*;*
(iii) $\overleftarrow{\mathbf{A}}(X) = 0.8185 > 0.8166 = \overleftarrow{\mathbf{A}}(Y)$*, then* $Y \preceq_{\overleftarrow{\mathbf{A}}} X$*.*

Definition 7.2.2. *Let* $A\colon \mathbb{U} \to [0,1]$ *be an admissible interleaving. The pseudo-inverse of* A *is the function* $A^{(-1)}\colon [0,1] \to \mathbb{U}$ *defined as follows*

$$A^{(-1)}(x) = \inf\{X \in \mathbb{U} : A(X) \geq x\}, \tag{108}$$

where the infimum in Eq.(108) is w.r.t. the admissible order $\preceq_A$*.*

Example 7.2.4. *See, in the following, illustrations considering the application of functions introduced in Definition 7.2.2.*

A. Taking $x = 0.9819$ *we observe that:*

1. $\mathbf{A}^{(-1)}(x) = [0.9, 0.9]$ *since* $\mathbf{A}([0.9, 0.9]) = 0.99 > 0.9819$ *and if* $\mathbf{A}(X) > 0.9819$ *for some* $X \in \mathbb{U}$ *then* $\underline{X} \geq 0.9$ *and therefore* $[0.9, 0.9] \leq X$*. So because* $\preceq_{\mathbf{A}}$ *is admissible then* $[0.9, 0.9] \preceq_{\mathbf{A}} X$*. Analogously, the left-reverse construction is given as* $\overleftarrow{\mathbf{A}}^{(-1)}(x) = [0.89, 0.91]$*;*
2. $\overleftarrow{\mathbf{A}} \circ \overleftarrow{\mathbf{A}}^{(-1)}(x) = \overleftarrow{\mathbf{A}}([0.89, 0.91]) = 0.9819$*, therefore* $\overleftarrow{\mathbf{A}} \circ \overleftarrow{\mathbf{A}}^{(-1)}(x) = x$*;*
3. $\mathbf{A} \circ \mathbf{A}^{(-1)}(x) = \mathbf{A}([0.9, 0.9]) = 0.99$ *and so,* $\mathbf{A} \circ \mathbf{A}^{(-1)}(x) > x$*;*
4. $\overleftarrow{\mathbf{A}} \circ \mathbf{A}^{(-1)}(x) = \overleftarrow{\mathbf{A}}([0.9, 0.9]) = 0.99$ *and so,* $\overleftarrow{\mathbf{A}} \circ \mathbf{A}^{(-1)}(x) > x$*;*
5. $\mathbf{A} \circ \overleftarrow{\mathbf{A}}^{(-1)}(x) = \mathbf{A}([0.89, 0.91]) = 0.8991$ *and so,* $\mathbf{A} \circ \overleftarrow{\mathbf{A}}^{(-1)}(x) < x$*;*

B. Now, if $z = 0.135$*, then* $\mathbf{A}^{(-1)}(z) = [0.15, 0.3]$*. And* $\overleftarrow{\mathbf{A}}^{(-1)}(z) = [0, 0.2]$ *since* $\overleftarrow{\mathbf{A}}([0, 0.2]) = 0.2 > 0.135$ *and if* $\overleftarrow{\mathbf{A}}(X) \geq 0.135$ *for some* $X \in [0,1]$ *then* $\overline{X} > 0.1$ *and* $\underline{X} \leq \overline{X}$*. So,* $[0, 0.2]) \leq X$*. Because* $\preceq_{\mathbf{A}}$ *is admissible then* $[0, 0.2] \preceq_{\mathbf{A}} X$*. And, other observations can be described below.*

1. $\mathbf{A} \circ \mathbf{A}^{(-1)}(z) = \mathbf{A}([0.15, 0.3]) = 0.135$ *so, it implies that* $\mathbf{A} \circ \mathbf{A}^{(-1)}(z) = z$*;*
2. $\overleftarrow{\mathbf{A}} \circ \overleftarrow{\mathbf{A}}^{(-1)}(z) = \overleftarrow{\mathbf{A}}([0, 0.2]) = 0.2$ *meaning that* $\overleftarrow{\mathbf{A}} \circ \overleftarrow{\mathbf{A}}^{(-1)}(z) > z$*.*
3. $\overleftarrow{\mathbf{A}} \circ \mathbf{A}^{(-1)}(z) = \overleftarrow{\mathbf{A}}([0.15, 0.3]) = 0.3105$ *meaning that* $\overleftarrow{\mathbf{A}} \circ \mathbf{A}^{(-1)}(z) > z$*;*
4. $\mathbf{A} \circ \overleftarrow{\mathbf{A}}^{(-1)}(z) = \mathbf{A}([0, 0.2]) = 0.02$ *meaning that* $\mathbf{A} \circ \overleftarrow{\mathbf{A}}^{(-1)}(z) < z$*.*

C. Taking $X = [0.8, 1.0]$*, it holds that*

1. $\mathbf{A}^{(-1)} \circ \mathbf{A}(X) = \mathbf{A}^{(-1)}(0.89\widetilde{09}) = [0.8, 1.0]$ *so,* $\mathbf{A}^{(-1)}(\mathbf{A}(X)) = X$*;*
2. $\overleftarrow{\mathbf{A}}^{(-1)} \circ \overleftarrow{\mathbf{A}}(X) = \overleftarrow{\mathbf{A}}^{(-1)}(0.98\widetilde{90}) = [0.8, 1.0]$ *and* $\overleftarrow{\mathbf{A}}^{(-1)}(\overleftarrow{\mathbf{A}}(X)) = X$*.*

3. $\mathbf{A}^{(-1)} \circ \overleftarrow{\mathbf{A}}(X) = \mathbf{A}^{(-1)}(0.989\tilde{90}) = [0.9, 0.9]$. *So,* $\mathbf{A}^{(-1)}(\overleftarrow{\mathbf{A}}(X)) \succ_{\mathbf{A}} X$*;*

4. $\overleftarrow{\mathbf{A}}^{(-1)} \circ \mathbf{A}(X) = \overleftarrow{\mathbf{A}}^{(-1)}(0.89\widetilde{09}) = [0, 0.9]$. *So,* $\overleftarrow{\mathbf{A}}^{(-1)}(\mathbf{A}(X)) \prec_{\mathbf{A}} X$

D. And, when $Z = [0.3, 0.6]$*, it holds that*

1. $\mathbf{A}^{(-1)} \circ \mathbf{A}(Z) = \mathbf{A}^{(-1)}(0.36) = [0.3, 0.6]$ *so, it implies that* $\mathbf{A}^{(-1)} \circ \mathbf{A}(Z) = Z$*;*

2. $\overleftarrow{\mathbf{A}}^{(-1)} \circ \overleftarrow{\mathbf{A}}(Z) = \overleftarrow{\mathbf{A}}^{(-1)}(0.63) = [0.3, 0.6]$ *meaning that* $\overleftarrow{\mathbf{A}}^{(-1)} \circ \overleftarrow{\mathbf{A}}(Z) = Z$*.*

3. $\mathbf{A}^{(-1)} \circ \overleftarrow{\mathbf{A}}(Z) = \mathbf{A}^{(-1)}(0.63) = [0.6, 0.6]$*, and* $\mathbf{A}^{(-1)}(\overleftarrow{\mathbf{A}}(Z)) \succ_{\mathbf{A}} Z$*;*

4. $\overleftarrow{\mathbf{A}}^{(-1)} \circ \mathbf{A}(Z) = \overleftarrow{\mathbf{A}}^{(-1)}(0.36) = [0, 0.4]$*, and* $\overleftarrow{\mathbf{A}}^{(-1)}(\mathbf{A}(Z)) \prec_{\mathbf{A}} Z$*.*

Additionally, one can easily observe the following comparisons:

1. *By A and B, if* $x > z$*,* $\mathbf{A} \circ \mathbf{A}^{(-1)}(x) = 0.99 > x > z = 0.135 = \mathbf{A} \circ \mathbf{A}^{(-1)}(z)$*;*

2. *By C and D, if* $X \succ_{\mathbf{A}} Z$*,* $\mathbf{A}^{(-1)}(\overleftarrow{\mathbf{A}}(X)) = [0.9, 0.9] \succ_{\mathbf{A}} [0, 0.9] = \overleftarrow{\mathbf{A}}^{(-1)}(\mathbf{A}(X))$*.*

The above examples motivate the analysis of the following properties of an injective and increasing aggregation function A, and its reverse construction is given in Eq.(108).

Lemma 1. *Let* $A : \mathbb{U} \to [0, 1]$ *be an admissible interleaving. The function* $A^{(-1)} : [0, 1] \to \mathbb{U}$ *defined in Eq. (108) verifies the following conditions:*

(1) $A^{(-1)}(A(X)) = X$*, for each* $X \in \mathbb{U}$*;*

(2) $x \leq A(A^{(-1)}(x))$*, for each* $x \in [0, 1]$*;*

(3) $x \leq y \Rightarrow A^{(-1)}(x) \preceq_A A^{(-1)}(y)$*, for* $x, y \in [0, 1]$*;*

(4) $A^{(-1)}(x) = \mathbf{0} \Leftrightarrow x = 0$ *and* $A^{(-1)}(x) = \mathbf{1} \Leftrightarrow x = 1$*, for each* $x \in [0, 1]$*;*

(5) $X \preceq_A Y \Rightarrow A^{(-1)}(A(X)) \preceq_A A^{(-1)}(A(Y))$*, for* $X, Y \in \mathbb{U}$*;*

(6) $x \leq y \Rightarrow A(A^{(-1)}(x)) \leq A(A^{(-1)}(y))$*, for* $x, y \in [0, 1]$*.*

Proof. (1), (4) and (5) are straightforward. And, for each $x, y \in [0, 1]$, it holds that: (2) By Eq.(108), $A^{(-1)}(x) = \inf\{X \in \mathbb{U} : A(X) \geq x\}$ and since A is an injective and increasing function w.r.t. the usual order on $[0, 1]$, we have that $A(A^{(-1)}(x)) = A(\inf\{X \in \mathbb{U} : A(X) \geq x\}) \geq x$.
(3) When $x \leq y$, then for each $X \in \mathbb{U}$, since $A(X) \geq y \Rightarrow A(X) \geq x$, then $\{X \in \mathbb{U} : A(X) \geq y\} \supseteq \{X \in \mathbb{U} : A(X) \geq x\}$. So, $\inf\{X \in \mathbb{U} : A(X) \geq y\} \preceq_A \inf\{X \in \mathbb{U} : A(X) \geq x\}$. Therefore, $A^{(-1)}(x) \preceq_A A^{(-1)}(y)$.
(6) Straightforward from item (3).
Therefore, Lemma 1 is verified. □

Now, we analyze other properties for the aggregation functions $\mathbf{A}, \overleftarrow{\mathbf{A}}$ and their reverse constructions. For that, firstly consider the following expressions related to a decimal expansion of a real number $x \in [0,1]$.

Definition 7.2.3. *Let* $x \in [0,1]$ *given as* $x = 0.x^{[1]}x^{[2]} \ldots x^{[2i]}x^{[2i+1]} \ldots x^{[2n-1]}x^{[2n]} \ldots$*.*

(i) $x \in [0,1]$ *is named pre-sequence when* $x^{[2i-1]} = x^{[2i]}$*,* $\forall i \in \mathbb{N}_j = \{1, \ldots, j\}$ *and* $x^{[2j+1]} < x^{[2j+2]}$ *for* $j \in \mathbb{N}$*;*

(ii) $x \in [0,1]$ *is named pos-sequence when* $x^{[2i-1]} = x^{[2i]}$*,* $\forall i \in \mathbb{N}_j$ *and* $x^{[2j+1]} > x^{[2j+2]}$ *for* $j \in \mathbb{N}$*.*

Example 7.2.5. *Illustrating Definition 7.2.3, consider the following two examples:*

- *Let* $x = 0.3333105 \in [0,1]$*. It illustrates a pos-sequence, since for* $j = 2$*,* $i \in \mathbb{N}_4$ *and* $x^{[1]} = x^{[2]} = x^{[3]} = x^{[4]} = 3$*. Moreover, we have that* $x^{[2j+1]} = x^{[5]} = 1$ *and* $x^{[2j+2]} = x^{[6]} = 0$*, meaning that* $x^{[2j+1]} > x^{[2j+2]}$*. So, by Definition 7.2.3,* $\mathbf{A}^{(-1)}(x) = [0.331, 0.331]$ *and* $\overleftarrow{\mathbf{A}}^{(-1)}(x) = [0.33, 0.3315]$*.*
- *Let* $x = 0.3333145 \in [0,1]$*. It exemplifies a pre-sequence, since for* $j = 2$ *and* $i \in \mathbb{N}_4$*, it holds that* $x^{[1]} = x^{[2]} = x^{[3]} = x^{[4]} = 3$*. However, we have that* $x^{[2j+1]} = x^{[5]} = 1$ *and* $x^{[2j+2]} = x^{[6]} = 4$*, meaning that* $x^{[2j+1]} < x^{[2j+2]}$*. So, by Definition 7.2.3,* $\mathbf{A}^{(-1)}(x) = [0.3315, 0.334]$ *and* $\overleftarrow{\mathbf{A}}^{(-1)}(x) = [0.33, 0.332]$*.*

The above constructions are formalized in the next lemma.

Lemma 2. *Let* $x \in [0,1]$ *expressed by Definition 7.2.3.*
(I) When $x \in [0,1]$ *is a pos-sequence then*

(1) $\mathbf{A}^{(-1)}(x) = \left[0.x^{[1]} \ldots x^{[2j-1]}x^{[2j+1]}, 0.x^{[2]} \ldots x^{[2j]}x^{[2j+1]}\right]$*;*

(2) $\mathbf{A}(\mathbf{A}^{(-1)}(x)) > x$*;*

(3) $\overleftarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}^{(-1)}(x)) = x$*;*

(4) $\mathbf{A}(\overleftarrow{\mathbf{A}}^{(-1)}(x)) < x$*.*

(II) When $x \in [0,1]$ *is a pre-sequence*

(5) $\overleftarrow{\mathbf{A}}^{(-1)}(x) = [0.x^{[1]} \ldots x^{[2j-1]}, 0.x^{[2]} \ldots x^{[2j]}x^{[2j+1]} + 1]$*.*

(6) $\mathbf{A}(\mathbf{A}^{(-1)}(x)) = x$*;*

(7) $\mathbf{A}(\overleftarrow{\mathbf{A}}^{(-1)}(x)) = \overleftarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}^{(-1)}(x)) > x$

Proof. Let $x \in [0,1]$ given as $x = 0.x^{[1]}x^{[2]} \ldots x^{[2j+1]}x^{[2j+2]} \ldots$. Based on Definition 7.2.3, the following holds:
(1) Since $x \in [0,1]$ is a pos-sequence, we have that $x^{[2j+1]} > x^{[2j+2]}$ and if $y = \mathbf{A}(X) \geq x$

for some $X \in \mathbb{U}$ then, because $\mathbf{A}(X)$ is pre-sequence, $A(X) > x$ and there is $i \in \mathbb{N}_j$ such that $y^{[2i]} > x^{[2i]}$ and $y^{[l]} = x^{[l]}$ for each $l < 2i$. So,the following is verified:

$$\mathbf{A}^{(-1)}(x) = \inf\{X \in \mathbb{U} : \mathbf{A}(X) \geq x\} = \left[0.x^{[1]}x^{[3]}\ldots x^{[2j-1]}x^{[2j+1]}, 0.x^{[2]}x^{[4]}\ldots x^{[2j]}x^{[2j+1]}\right] \quad (109)$$

(2) Straightforward from item (1).
(3) Since x is a pos-sequence, $x^{[2i-1]} = x^{[2i]}$, $\forall i \in \mathbb{N}_j = \{1,\ldots,j\}$ and $x^{[2j+1]} > x^{[2j+2]}$, for $j \in \mathbb{N}$. Then, $\overleftarrow{\mathbf{A}}^{(-1)}(x) = [0.x^{[2]}x^{[4]}\ldots x^{[2j+2]}\ldots, 0.x^{[1]}x^{[3]}\ldots x^{[2j+1]}\ldots]$. So, $\overleftarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}^{(-1)}(x)) = 0.x^{[1]}x^{[2]}\ldots x^{[2j+1]}x^{[2j+2]}\ldots = x$.
(4) Straightforward from item (1).
(5) Since $x \in [0,1]$ is a pre-sequence, we have that $x^{[2j+1]} < x^{[2j+2]}$ and if $y = \overleftarrow{\mathbf{A}}(X) \geq x$ for some $X \in \mathbb{U}$ then, because $\overleftarrow{\mathbf{A}}(X)$ is pos-sequence, $A(X) > x$ and there is $i \in \mathbb{N}_j$ such that $y^{[2i+1]} > x^{[2i+1]}$ and $y^{[l]} = x^{[l]}$ for each $l \leq 2i$. So, the following holds:

$$\overleftarrow{\mathbf{A}}^{(-1)}(x) = \inf\{X \in \mathbb{U} : \overleftarrow{\mathbf{A}}(X) \geq x\} = \left[0.x^{[1]}x^{[3]}\ldots x^{[2j-1]}, 0.x^{[2]}x^{[4]}\ldots x^{[2j]}x^{[2j+1]} + 1\right] \quad (110)$$

(6) Since $\mathbf{A}^{(-1)}$ and $\mathbf{A}$ are aggregations and x is a pre-sequence, meaning that $x^{[2i-1]} = x^{[2i]}$, $\forall i \in \mathbb{N}_j$ and $x^{[2j+2]} > x^{[2j+1]}$, for $j \in \mathbb{N}$. Then, we have that $\mathbf{A}^{(-1)}(x) = [0.x^{[1]}x^{[3]}\ldots x^{[2j-1]}x^{[2j+1]}\ldots, 0.x^{[2]}x^{[4]}\ldots x^{[2j]}x^{[2j+2]}\ldots]$ and it results that $\mathbf{A}(\mathbf{A}^{(-1)}(x)) = 0.x^{[1]}x^{[2]}\ldots x^{[2j+1]}x^{[2j+2]}\ldots = x$.
(7) Straightforward from item (5).
Therefore, Lemma 2 is verified. □

7.3 Fuzzy Connectives on $\langle \mathbb{U}, \preceq_A \rangle$

This section explores the notion of admissible interleaving orders in the definition of the width-based interval-valued extension of fuzzy negations, aggregations and restricted equivalence (dissimilarity) functions on $\langle \mathbb{U}, \preceq_A \rangle$.

7.3.1 Negations on $\langle \mathbb{U}, \preceq_A \rangle$

Now we consider the study of interval-valued fuzzy negation on $\langle \mathbb{U}, \preceq_A \rangle$.

A. Generating Negation by Injective and Increasing Function

Theorem 7.3.1. *Let $A : \mathbb{U} \to [0,1]$ be an injective and increasing function and $N : [0,1] \to [0,1]$ be a strict fuzzy negation. The function $\mathbb{N}^A : \mathbb{U} \to \mathbb{U}$ defined by*

$$\mathbb{N}^A(X) = A^{(-1)}(N(A(X))); \quad (111)$$

is a $\langle \mathbb{U}, \preceq_A \rangle$-negation, which is called a representable $\langle \mathbb{U}, \preceq_A \rangle$-negation.

Proof. Straightforward from Corollary 7.2.1 and Lemma 1. □

Proposition 7.3.1. *Let A be an admissible interleaving. Whenever $N\colon [0,1] \to [0,1]$ is a strong fuzzy negation, $\mathbb{N}^A$ verifies $\mathbb{N}^A(\mathbb{N}^A(X)) \succeq_A X$.*

Proof. Let $N\colon [0,1] \to [0,1]$ be a strong fuzzy negation. The following holds:

$$\begin{aligned}\mathbb{N}^A(\mathbb{N}^A(X)) &= A^{(-1)}(N(A(A^{(-1)}(N(A(X)))))) \text{ by Eq.(111)}\\ &\geq A^{(-1)}(N(N(A(X))))\text{by Lemma 1 (item 2.)}\\ &= A^{(-1)}(A(X)) = X \text{ by } \mathbb{N}3 \text{ and Lemma 1 (item 1).}\end{aligned}$$

Then, because $\preceq_{\mathbf{A}}$ is an admissible order, $\mathbb{N}^A(\mathbb{N}^A(X)) \succeq_A X$. So, Prop. 7.3.1 holds. □

B. Generating Negation on $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$ by DDI functions

We start this subsection by illustrating the results from Theorem 7.3.1.

Example 7.3.1. *The $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$-representable negation generated by the standard negation N_S is given as:*

$$\mathbb{N}_S^{\mathbf{A}}(X) = \mathbf{A}^{(-1)}(N_S(\mathbf{A}(X))), \forall X \in \mathbb{U}. \tag{112}$$

By Example 7.2.3, taking $X = [0.15, 0.88] \in \mathbb{U}$, then we obtain that $\mathbb{N}_S^{\mathbf{A}}(X) = \mathbf{A}^{(-1)}(N_S(0.1858)) = \mathbf{A}^{(-1)}(0.8142) = [0.8, 0.8]$. And, it holds that $\mathbb{N}_S^{\mathbf{A}}([0.8, 0.8]) = \mathbf{A}^{(-1)}(0.12) = [0.1, 0.2]$. So, $\mathbb{N}_S^{\mathbf{A}}(\mathbb{N}_S^{\mathbf{A}}(X)) \preceq_{\mathbf{A}} X$. Analogously, for $Y = [0.26, 0.43] \in \mathbb{U}$, $\mathbb{N}_S^{\mathbf{A}}(Y) = \mathbf{A}^{(-1)}(N_S(0.2463)) = \mathbf{A}^{(-1)}(0.7537) = [0.7, 0.7]$. Therefore, $\mathbb{N}_S^{\mathbf{A}}(\mathbb{N}_S^{\mathbf{A}}(Y)) \preceq_{\mathbf{A}} Y$, since ${\mathbb{N}_S}^{\mathbf{A}}([0.7, 0.7]) = [0.2, 0.3]$. So, $\mathbb{N}_S^{\mathbf{A}}(\mathbb{N}_S^{\mathbf{A}}(Y)) \preceq_{\mathbf{A}} Y$.

Corollary 7.3.1. *Let $x \in [0,1]$ expressed by Definition 7.2.3 and $\preceq_{\mathbf{A}}$ be the admissible order in Corollary 7.2.1. Then, it holds that :*

(i) If $x \in N(\mathbf{A}[\mathbb{U}])$ then x is a pos-sequence;

(ii) If $x \in \mathbf{A}[\mathbb{U}]$ then x is a pre-sequence.

Proof. Straightforward. □

Now, the concept of interleaving fuzzy negation is presented, based on the admissible pairwise $(\preceq_{\mathbf{A}}, \preceq_{\overleftarrow{\mathbf{A}}})$-order.

Theorem 7.3.2. *Let $N_S\colon [0,1] \to [0,1]$ be the standard negation. The function $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}\colon \mathbb{U} \to \mathbb{U}$ defined by*

$$\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(X))) \tag{113}$$

is called the N_S-interleaving negation on $\mathbb{U}$ w.r.t. admissible pairwise $(\preceq_{\mathbf{A}}, \preceq_{\overleftarrow{\mathbf{A}}})$-order and it satisfies the following properties:

1. *$\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \mathbf{0}$ iff $X = \mathbf{1}$;*

2. *$\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \mathbf{1}$ iff $X = \mathbf{0}$;*

3. *If $X \prec_{\mathbf{A}} Y$ then $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(Y) \prec_{\overleftarrow{\mathbf{A}}} \mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X)$;*

4. *$\mathbb{N}_S^{\mathbf{A}}(X) = \mathbb{N}_S^{\mathbf{A}}(Y)$ iff $X = Y$, for each $X \in \mathbb{U}$.*

Proof. (1). Let $X \in \mathbb{U}$, given as $X = \left[0.\underline{X}^{[1]}\underline{X}^{[2]}\ldots\underline{X}^{[n]}\ldots, 0.\overline{X}^{[1]}\overline{X}^{[2]}\ldots\overline{X}^{[n]}\ldots\right]$, and so, there exists $i \in \mathbb{N}$ such that $\underline{X}^{[j]} = \overline{X}^{[j]}$ for each $j \leq i$ and $\underline{X}^{[i+1]} < \overline{X}^{[i+1]}$. So, by application of N_S, we obtain that

$$
\begin{aligned}
&\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(X)))\\
&= \overleftarrow{\mathbf{A}}^{(-1)}\left(N_S\left(0.\underline{X}^{[1]}\overline{X}^{[1]}\ldots\underline{X}^{[i]}\overline{X}^{[i]}\underline{X}^{[i+1]}\overline{X}^{[i+1]}\ldots\underline{X}^{[n]}\overline{X}^{[n]}\ldots\right)\right)\\
&= \overleftarrow{\mathbf{A}}^{(-1)}\Big(0.(9-\underline{X}^{[1]})(9-\overline{X}^{[1]})\ldots(9-\underline{X}^{[i]})(9-\overline{X}^{[i]})\,(9-\underline{X}^{[i+1]})(9-\overline{X}^{[i+1]})(9-\underline{X}^{[i+1]})\\
&\qquad\qquad\ldots(9-\underline{X}^{[n]})(9-\overline{X}^{[n]})\ldots\Big)\\
&= \Big[0.(9-\overline{X}^{[1]})\ldots(9-\overline{X}^{[i]})(9-\overline{X}^{[i+1]})\ldots\overline{X}^{[n]}), 0.(9-\underline{X}^{[1]})\ldots(9-\underline{X}^{[i]})(9-\underline{X}^{[i+1]})\ldots\underline{X}^{[n]})\Big]
\end{aligned}
$$

Since $(9-\underline{X}^{[j]}) = (9-\overline{X}^{[j]})$, for $j \leq i$ and $(9-\underline{X}^{[i+1]}) > (9-\overline{X}^{[i+1]})$, then

$$
\begin{aligned}
\underline{\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X)} &= 0.(9-\overline{X}^{[1]})\ldots(9-\overline{X}^{[i]})(9-\overline{X}^{[i+1]})\ldots(9-\overline{X}^{[n]})\\
&\leq 0.(9-\underline{X}^{[1]})\ldots(9-\underline{X}^{[i]})(9-\underline{X}^{[i+1]})\ldots(9-\underline{X}^{[n]})\ldots = \overline{\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X)}
\end{aligned}
$$

Therefore, $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(X)))$ is well defined. (2.) In addition, the boundary conditions are also verified:

(i) $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(\mathbf{0}) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(\mathbf{0})))) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(0)) = \overleftarrow{\mathbf{A}}^{(-1)}(1) = \mathbf{1}$. If $\mathbf{0} \prec_{\mathbf{A}} X$ then $\mathbf{A}(X) > 0$ and therefore $N_S(\mathbf{A}(X)) < 1$. Since, $\mathbf{A}(X)$ is a pre-sequence then, by Corollary 7.3.1, $N_S(\mathbf{A}(X))$ is a pos-sequence. Hence, by Lemma 2 (1), $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) \neq \mathbf{0}$. Therefore, $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \mathbf{0}$ iff $X = \mathbf{1}$.

(ii) $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(\mathbf{1}) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(\mathbf{1})))) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(1)) = \overleftarrow{\mathbf{A}}^{(-1)}(0) = \mathbf{0}$. If $X \prec_{\mathbf{A}} \mathbf{1}$ then $\mathbf{A}(X) < 1$ and therefore $N_S(\mathbf{A}(X)) > 0$. Since, $\mathbf{A}(X)$ is a pre-sequence then, by Corollary 7.3.1, $N_S(\mathbf{A}(X))$ is a pos-sequence. Hence, by Lemma 2 (1), $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) \neq \nVdash$. Therefore, $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \mathbf{1}$ iff $X = \mathbf{0}$.

(3.) Moreover, let $X, Y \in \mathbb{U}$. If $X \preceq_{\mathbf{A}} Y$ then $\mathbf{A}(X) \leq \mathbf{A}(Y)$ and therefore $N_S(\mathbf{A}(X)) \geq N_S(\mathbf{A}(Y))$. So, by Lemma 1(3), we have that $\overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(Y))) \preceq_{\overleftarrow{\mathbf{A}}} \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(X)))$.

Thereby, $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(Y) \prec_{\overleftarrow{\mathbf{A}}} \mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X)$.
(4.) Now, by Corollary 7.3.1, if $x \in N_S(\mathbf{A}(X))$ implies that X is a pos-sequence. Then, see the following results:

$$\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(Y) \Leftrightarrow \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(X))) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(Y))) \text{ by Eq.(113)}$$
$$\Leftrightarrow \overleftarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}^{(-1)}(N_S(\mathbf{A}(X)))) = \overleftarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}^{(-1)}((N_S(\mathbf{A}(Y))))) \text{ because } \overleftarrow{\mathbf{A}} \text{ is injectiva}$$
$$\Leftrightarrow N_S(\mathbf{A}(X)) = N_S(\mathbf{A}(Y)) \text{ by Corollary 7.3.1 and Lemma 2(5)}$$
$$\Leftrightarrow \mathbf{A}(X) = \mathbf{A}(Y) \Leftrightarrow X = Y \text{ because } N_S \text{ is strong.}$$

Concluding, Theorem 7.3.2 is verified. □

Example 7.3.2. *Illustrating the $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$-negation presented in Eq.(113):*

- *If* $X = [0.15, 0.88] \in \mathbb{U}$, *then we have that* $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(0.1858)) = \overleftarrow{\mathbf{A}}^{(-1)}(0.8141\tilde{9}) = [0.12, 0.85]$*; In addition,* $w(X) = 0.73 = w(\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X))$.
- *And, if* $Y = [0.26, 0.43]$, $\mathbb{N}_S{}^{\overleftarrow{\mathbf{A}}}(Y) = \overleftarrow{\mathbf{A}}^{(-1)}(N_S(0.2463)) = \overleftarrow{\mathbf{A}}^{(-1)}(0.75369̃) = [0.57, 0.74]$. *In addition,* $w(Y) = 0.17 = w(\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(Y))$.

Therefore, we can observe that $X \not\leq_{\mathbb{U}} Y$ *and* $Y \not\leq_{\mathbb{U}} X$*, meaning that they are not comparable in* $\langle \mathbb{U}, \leq_{\mathbb{U}} \rangle$*. Moreover,* $X \prec_{\mathbf{A}} Y$ *implies that* $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(Y) \prec_{\overleftarrow{\mathbf{A}}} \mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X)$*. However, it also implies that* $\mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(X) \prec_{\mathbf{A}} \mathbb{N}_S^{\overleftarrow{\mathbf{A}}}(Y)$.

C. Generating Negation on $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$ by Equilibrium Point

To introduce an expression to obtain interval-valued fuzzy negation w.r.t. an admissible interleaving order we introduce fuzzy negation N_{e}, which has e as the equilibrium point.

Lemma 3. *Let* $\mathrm{e} \in (0,1)$*. Then,* $N_{\mathrm{e}} : [0,1] \to [0,1]$ *given as*

$$N_{\mathrm{e}}(x) = \begin{cases} 1 - \frac{(1-\mathrm{e})}{\mathrm{e}}x, & \text{if } x \leq e, \\ \frac{\mathrm{e}}{1-\mathrm{e}}(1-x), & \text{otherwise;} \end{cases} \tag{114}$$

is a strong (strict) fuzzy negation and it has e *as the equilibrium point.*

Remark 7.3.1. *Let* N_{e} *be the negation given by Eq.(114). Then, the following holds:*

(i) when $\mathrm{e} = 0.5$ *then* $N_{\mathrm{e}} = N_S$*; and*

(ii) when $\mathrm{e} = \phi(0.5)$ *then* $N_{\mathrm{e}}(x) = \phi^{-1}(N_S(\phi(x)))$*, for each automorphism* $\phi : U \to U$*, see (Bustince; Burillo; Soria, 2003b).*

Theorem 7.3.3. *Let $N_{\rm e}$ be the negation given by Eq.(114) which has $\rm e$ as the equilibrium point. The function $\mathbb{N}_{\rm e}^{\mathbf{A}}\colon \mathbb{U} \to \mathbb{U}$ defined by*

$$\mathbb{N}_{\rm e}^{\mathbf{A}}(X) = \begin{cases} \overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(\mathbf{A}(X))), & \textit{if } \ \underline{X} \leq e, \\ \mathbf{A}^{(-1)}(N_{\rm e}(\mathbf{A}(X))), & \textit{otherwise}; \end{cases} \tag{115}$$

is called as the N_e-interleaving negation on $\mathbb{U}$ w.r.t. admissible pairwise $(\preceq_{\mathbf{A}}, \preceq_{\overleftarrow{\mathbf{A}}})$-order and it satisfies the following properties:

1. $\mathbb{N}_{\rm e}^{\mathbf{A}}(\mathbf{1}) = \mathbf{0}$ *and* $\mathbb{N}_{\rm e}^{\mathbf{A}}(\mathbf{0}) = \mathbf{1}$*;*
2. *If $X \prec_{\mathbf{A}} Y$ then $\mathbb{N}_{\rm e}^{\mathbf{A}}(Y) \prec_{\overleftarrow{\mathbf{A}}} \mathbb{N}_{\rm e}^{\mathbf{A}}(X)$;*
3. $\mathbb{N}_{\rm e}^{\mathbf{A}}(X) = \mathbb{N}_{\rm e}^{\mathbf{A}}(Y)$ *iff $X = Y$, for each $X \in \mathbb{U}$;*
4. $\mathbb{N}_{\rm e}^{\mathbf{A}}$ *is strictly decreasing;*
5. $\mathbb{N}_{\rm e}^{\mathbf{A}}(E) = E$*, where $E = \mathbf{A}^{(-1)}(\rm e)$.*

Proof. Firstly, $\mathbb{N}_{\rm e}^{\mathbf{A}}(\mathbf{0}) = \overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(\mathbf{A}(\mathbf{0}))) = \overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(0)) = \overleftarrow{\mathbf{A}}^{(-1)}(1) = \mathbf{1}$ and $\mathbb{N}_{\rm e}(\mathbf{1}) = \overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(\mathbf{A}(\mathbf{1})) = \overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(1)) = \overleftarrow{\mathbf{A}}^{(-1)}(0) = \mathbf{0}$. If $X \prec_{\mathbf{A}} Y$ then, analogously to Theorem 7.3.2.3, $\mathbb{N}_{\rm e}^{\mathbf{A}}(Y) \prec_{\overleftarrow{\mathbf{A}}} \mathbb{N}_{\rm e}^{\mathbf{A}}(X)$. By Corollary 7.3.1, if $x \in N_{\rm e}(\mathbf{A}(X))$ implies that X is a pos-sequence. Then, it results that

$$\begin{aligned} \mathbb{N}_{\rm e}^{\mathbf{A}}(X) = \mathbb{N}_{\rm e}^{\mathbf{A}}(Y) &\Leftrightarrow \overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(\mathbf{A}(X))) = \overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(\mathbf{A}(Y))) \text{ by Eq.(111)} \\ &\Leftrightarrow \overleftarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}^{(-1)}(N_{\rm e}(\mathbf{A}(X)))) = \overleftarrow{\mathbf{A}}(\overleftarrow{\mathbf{A}}^{(-1)}((N_{\rm e}(\mathbf{A}(Y))))) \text{ because } \overleftarrow{\mathbf{A}} \text{ is injectiva} \\ &\Leftrightarrow N_{\rm e}(\mathbf{A}(X)) = N_{\rm e}(\mathbf{A}(Y)) \text{ by Corollary 7.3.1 and Lemma 2(5)} \\ &\Leftrightarrow \mathbf{A}(X) = \mathbf{A}(Y) \Leftrightarrow X = Y \text{ because } N_e \text{ is strong.} \end{aligned}$$

In addition, from previous two item, straightforward $\mathbb{N}_{\rm e}^{\mathbf{A}}$ is strictly decreasing. And finally, since $E = \mathbf{A}^{(-1)}(\rm e)$ then we obtain that $\mathbb{N}_{\rm e}^{\mathbf{A}}(E) = \overleftarrow{\mathbf{A}}^{(-1)}N_{\rm e}(\mathbf{A}(E)) = \overleftarrow{\mathbf{A}}^{(-1)}N_{\rm e}(\mathbf{A}(\mathbf{A}^{(-1)}(\rm e))) = \overleftarrow{\mathbf{A}}^{(-1)}N_{\rm e}(\rm e) = \overleftarrow{\mathbf{A}}^{(-1)}(\rm e) = E = [0, \rm e]$. Therefore, we also proved that $N_{\rm e}(\overleftarrow{\mathbf{A}}(E)) = E$. □

Example 7.3.3. *Let $N_{\rm e} : [0,1] \to [0,1]$ given in Eq. (114) with the equilibrium point as $\rm e = \frac{4}{5}$. By Eq.(111), the function $\mathbb{N}_{\frac{4}{5}}^{\mathbf{A}}\colon \mathbb{U} \to \mathbb{U}$ given as*

$$\mathbb{N}_{\frac{4}{5}}^{\mathbf{A}}(X) = \begin{cases} \overleftarrow{\mathbf{A}}^{(-1)}(1 - \frac{1}{4}(\mathbf{A}(X))), & \textit{if } \ \underline{X} \leq \frac{4}{5}, \\ \mathbf{A}^{(-1)}(4(1 - \mathbf{A}(X)), & \textit{otherwise}; \end{cases} \tag{116}$$

is a strict IvFN w.r.t admissible $\preceq_{\mathbf{A}}$-order. In addition, it also has $E = \left[0, \frac{4}{5}\right]$ as the equilibrium point.

7.3.2 Aggregations on $\langle \mathbb{U}, \preceq_A \rangle$

Proposition 7.3.2. *Let $M\colon [0,1]^n \to [0,1]$ be (a strictly increasing) aggregator, and $A\colon \mathbb{U} \to [0,1]$ be an admissible interleaving function. Then, $\mathbb{M}^A\colon \mathbb{U}^n \to \mathbb{U}$, given as*

$$\mathbb{M}^A(X_1,\ldots,X_n) = A^{(-1)}\left(M(A(X_1),\ldots,A(X_n))\right) \tag{117}$$

is an IvA function related to the admissible $\preceq_A$-order. In addition, if M is idempotent, then $\mathbb{M}^A$ is also idempotent.

Proof. Let $M\colon [0,1]^n \to [0,1]$ be a strictly increasing aggregation function and $A\colon \mathbb{U} \to [0,1]$ be an admissible interleaving. So, the following holds:

$$\mathbb{M}^A(\mathbf{0},\ldots,\mathbf{0}) = A^{(-1)}\left(M(A(\mathbf{0}),\ldots,A(\mathbf{0}))\right) = A^{(-1)}\left(M(0,\ldots,0)\right) = A^{(-1)}(0) = \mathbf{0};$$
$$\mathbb{M}^A(\mathbf{1},\ldots,\mathbf{1}) = A^{(-1)}\left(M(A(\mathbf{1}),\ldots,A(\mathbf{1}))\right) = A^{(-1)}\left(M(1,\ldots,1)\right) = A^{(-1)}(1) = \mathbf{1}.$$

If $A(X_i) \leq A(Y_i)$, $\forall i \in \mathbb{N}_n$, then $X_i \preceq_A Y_i, \forall i \in \mathbb{N}_n$ and follows that

$$M(A(X_1),\ldots,A(X_n)) \leq M(A(Y_1),\ldots,A(Y_n)).$$

So, since A is an increasing function,

$$A^{(-1)}(M(A(X_1),\ldots,A(X_n))) \preceq_A A^{(-1)}(M(A(Y_1),\ldots,A(Y_n))).$$

Therefore $\mathbb{M}^A(X_1,\ldots,X_n) \preceq_{\mathbf{A}} \mathbb{M}^A(Y_1,\ldots,Y_n)$. Moreover, when M is idempotent aggregation, by Lemma 1(1), we have that:

$$\mathbb{M}^A(X,X,\ldots,X) = A^{(-1)}\left(M(A(X),\ldots,A(X)\right) = A^{(-1)}(A(X)) = X.$$

Thus, Proposition 7.3.2 is verified. □

Example 7.3.4. *Let $A\colon \mathbb{U} \to [0,1]$ be an admissible interleaving. When M is the minimum, then $\mathbb{M}^A$ also is the minimum but w.r.t. $\preceq_A$. So, by Proposition7.3.2,*

$$\mathbb{M}^A(X_1,\ldots,X_n){=}A^{(-1)}(M(A(X_1),\ldots,A(X_n))){=}A^{(-1)}M_{i=1}^{n}A(X_i).$$

Analogous expressions can be obtained for the idempotent aggregations as arithmetic means, maximum, left-median, or right-median. So, $\mathbb{M}^{\mathbf{A}}$ is an idempotent IvA function related to the admissible $\preceq_A$-order.

Proposition 7.3.3. *Let M be a strict aggregation function and A an admissible interleaving. The $\mathbb{M}^A$ is idempotent if, and only if, $\mathbb{M}^A$ is an average function $\langle \mathbb{U}, \preceq_A \rangle$.*

Proof. Straightforward Proposition 7.3.2. □

7.3.3 Width-based Interval-valued Restricted Equivalence Functions

Methodologies to obtain REF operator by admissible $\preceq_A$-orders are presented.

Theorem 7.3.4. *Let $A : \mathbb{U} \to [0,1]$ be an admissible interleaving, and $\preceq_A$ be the order on $\mathbb{U}$ defined in Eq. (103). Then the function $\mathbb{S}_A : \mathbb{U}^2 \to \mathbb{U}$ defined by*

$$\mathbb{S}_A(X,Y) = [\min(a, 1-\omega(X), 1-\omega(Y)), \max(a, 1-|A(X)-A(Y)|)], \tag{118}$$

where $a = \min\left(\frac{A(X)}{A(Y)}, \frac{A(Y)}{A(X)}\right)$ with the convention that $\frac{x}{0} = 1$, is an interval-valued restricted equivalence function w.r.t. $\preceq_A$.

Proof. First observe that $\mathbb{S}_A$ is well defined, once $\min(a, 1-\omega(X), 1-\omega(Y)) \leq a \leq \max(a, 1-|A(X)-A(Y)|)$.
$\mathbb{S}1$: If $\mathbb{S}(X,Y) = \mathbf{0}$ then $\max(a, 1-|A(X)-A(Y)|) = 0$. So, $|A(X)-A(Y)| = 1$. So, by $(\mathbb{A}0)$ and $(\mathbb{A}1)$, we have that $\{X,Y\} = \{\mathbf{0},\mathbf{1}\}$. Conversely, $\mathbb{S}(\mathbf{1},\mathbf{0}) = [\min(\min(1,0),1,1), \max(\min(1,0), 1-|1-0|)] = \mathbf{0} = [\min(\min(0,1),1,1), \max(\min(0,1), 1-|0-1|)] = \mathbb{S}(\mathbf{0},\mathbf{1})$.
$\mathbb{S}2$: For each $X \in \mathbb{U}$, $\mathbb{S}(X,X) = [\min(a, 1-\omega(X), 1-\omega(X)), \max(a, 1-|A(X)-A(X)|)] = [\min(\min(1,1), 1-\omega(X)), \max(\min(1,1), 1-0] = [1-\omega(X), 1]$.
$\mathbb{S}3$: Direct from definition of $\mathbb{S}_A$.
$\mathbb{S}4$: Let $X,Y,Z \in \mathbb{U}$ such that $X \preceq_A Y \preceq_A Z$ and $\omega(X) = \omega(Y) = \omega(Z)$. Then, $A(X) \leq A(Y) \leq A(Z)$ and

$$a_1 = \min\left(\frac{A(X)}{A(Z)}, \frac{A(Z)}{A(X)}\right) = \frac{A(X)}{A(Z)} \leq \frac{A(X)}{A(Y)} = \min\left(\frac{A(X)}{A(Y)}, \frac{A(Y)}{A(X)}\right) = a_2.$$

And , $\min(a_1, 1-\omega(X), 1-\omega(Z)) \leq \min(a_2, 1-\omega(X), 1-\omega(Y))$ and $|A(X) - A(Z)| = A(Z) - A(X) \geq A(Y) - A(X) = |A(X) - A(Y)|$, i.e. $\max(a_1, 1-|A(x) - A(Z)|) \leq \max(a_2, 1-|A(x)-A(Y)|)$. Therefore, $\mathbb{S}_A(X,Z) = [\min(a_1, 1-\omega(X), 1-\omega(Z)), \max(a_1, 1-|A(x)-A(Z)|)] \leq [\min(a_2, 1-\omega(X), 1-\omega(Y)), \max(a_2, 1-|A(x) - A(Y)|)] = \mathbb{S}_A(X,Y)$. Analogously, we prove $\mathbb{S}_A(X,Z) \leq \mathbb{S}_A(Y,Z)$. Finally, once $\preceq_A$ is an admissible order then $\mathbb{S}_A(X,Z) \preceq_A \mathbb{S}_A(X,Y)$ and $\mathbb{S}_A(X,Z) \preceq_A \mathbb{S}_A(Y,Z)$. $\square$

Example 7.3.5. *In order to illustrate the method introduced in Theorem 7.3.4, consider the calculus to obtain $a_{XY} = \min\left(\frac{\mathbf{A}(X)}{\mathbf{A}(Y)}, \frac{\mathbf{A}(Y)}{\mathbf{A}(X)}\right)$ for any pair X and Y of intervals on $\mathbb{U}$. In particular, taking the interval-data: $X = [0.1, 0.3], Y = [0.5, 0.7], Z = [0.6, 0.8] \in \mathbb{U}$.*

$$a_{XY} = \frac{0.13}{0.57} = 0.2280;\ a_{YZ} = \frac{0.57}{0.68} = 0.838235294;\ a_{XZ} = \frac{0.13}{0.68} = 0.1911.$$

Observing that $\omega(X) = \omega(Y) = \omega(Z) = 0.2$ and $X \preceq_{\mathbf{A}} Y \preceq_{\mathbf{A}} Z$, the related ω-IvREF *w.r.t.* $\preceq_{\mathbf{A}}$ *are given as follows:*

$$\mathbb{S}_A(X,Y) = [\min(0.228, 0.8, 0.8), \max(0.228, 0.56)] = [0.228, 0.56]$$
$$\mathbb{S}_A(Y,Z) = [\min(0.838235294, 0.8, 0.8), \max(0.838235294, 0.89)] = [0.8, 0.89];$$
$$\mathbb{S}_A(X,Z) = [\min(0.1911, 0.8, 0.8), \max(0.1911, 0.45)] = [0.1911, 0.45].$$

Theorem 7.3.5. *Let* $N_{\mathrm{e}} : [0,1] \to [0,1]$ *given in Eq. (114) taking* e *as the equilibrium point. The function* $\mathbb{R}_{\mathbf{A}} : \mathbb{U}^2 \to \mathbb{U}$ *given as*

$$\mathbb{R}_{\mathbf{A}}(X,Y)=[\max(0, N_{\mathrm{e}}(|\mathbf{A}(X)-\mathbf{A}(Y)|)-\max(\omega(X),\omega(Y))), N_{\mathrm{e}}(|\mathbf{A}(X)-\mathbf{A}(Y)|)] \quad (119)$$

is an interval-valued restricted equivalence function w.r.t. the $\preceq_{\mathbf{A}}$*-order.*

Proof. Firstly, observe that $\mathbb{R}_{\mathbf{A}}$ is well defined, once $0 \leq \max(0, N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(Y)|) - \max(\omega(X), \omega(Y))) \leq N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(Y)|) \leq 1$.

$\mathbb{S}$1: If $\mathbb{R}_{\mathbf{A}}(X,Y) = \mathbf{0}$ then $N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(Y)|) = 0 \Rightarrow |\mathbf{A}(X) - \mathbf{A}(Y)| = 1 \Rightarrow X = 0 \textit{ and } Y = 1 \textit{ or } X = 1 \textit{ and } Y = 0$. Moreover, $\max(0, N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(Y)|) - \max(\omega(X), \omega(Y))) = \max(0, 0 - \max(\omega(X), \omega(Y))) = 0$. Now, conversely, taking $X = 0 \textit{ and } Y = 1 \textit{ or } X = 1 \textit{ and } Y = 0$, then $\mathbb{R}_{\mathbf{A}} = \max(0, N_{\mathrm{e}}(1) - \max(1, 0), N_{\mathrm{e}}(1)) = 0$.

$\mathbb{S}$2: For each $X \in \mathbb{U}$, we have that $\mathbb{R}_{\mathbf{A}}(X,X) = [\max(0, N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(X)|) - \max(\omega(X), \omega(X))), N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(X)|)]$. So, it means that $\mathbb{R}(X,X) = [\max(0, N_{\mathrm{e}}(0) - \max(\omega(X), \omega(Y))), N_{\mathrm{e}}(0)] = [1 - \omega(X), 1]$.

$\mathbb{S}$3: Direct from definition of $\mathbb{R}_{\mathbf{A}}$.

$\mathbb{S}$4: Let $X, Y, Z \in \mathbb{U}$ a such that $X \preceq_{\mathbf{A}} Y \preceq_{\mathbf{A}} Z$ and $\omega(X) = \omega(Y) = \omega(Z)$. Then, when $\mathbf{A}(X) \leq \mathbf{A}(Y) \leq \mathbf{A}(Z)$ we have that

$$\begin{aligned}\mathbb{R}_{\mathbf{A}}(X,Z) &= [\max(0, N_{\mathrm{e}}(|\mathbf{A}(X)-\mathbf{A}(Z)|)-\max(\omega(X),\omega(Z))), N_{\mathrm{e}}(|\mathbf{A}(X)-\mathbf{A}(Z)|)] \\ &\preceq_{\mathbf{A}} \max(0, N_{\mathrm{e}}(|\mathbf{A}(X)-\mathbf{A}(Y)|)-\max(\omega(X),\omega(Y))), N_{\mathrm{e}}(|\mathbf{A}(X)-\mathbf{A}(Y)|)] \\ &= \mathbb{R}_{\mathbf{A}}(X,Y)\end{aligned}$$

And, analogously, one can prove that $\mathbb{R}_{\mathbf{A}}(X,Z) \preceq_{\mathbf{A}} \mathbb{R}_{\mathbf{A}}(Y,Z)$.

Therefore, Theorem 7.3.4 is verified. □

7.4 Summary

This chapter considers the concepts of total admissible orders, $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$ to define the corresponding axiomatic expressions of interval-valued fuzzy connective. In particular, these definitions are illustrated w.r.t. Xu and Yager's $\preceq_{XY}$-order.

As a more relevant contribution, this chapter presents a new admissible order based on injective and increasing function A. This approach to total orders, as an extension of the usual partial $\leq_{\mathbb{U}}$-order on $\mathbb{U}$, does not depend on a pair of functions, as remains the literature (Xu; Yager, 2006b; Zapata et al., 2017; Bustince; Barrenechea; Pagola, 2008).

The main conditions under which the connectives were defined based on the lattice $\langle \mathbb{U}, \preceq_{\mathbb{U}} \rangle$ are also discussed in the proposed theorems.

Illustrating this methodology, the DDI functions are defined and the related admissible order $\preceq_{\mathbf{A}}$ is introduced, including its reverse construction and many examples stressing main properties and compositions.

Preceding the definition of width-based interval-valued entropy on $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$, the notion of width-based interval-valued restricted equivalence (dissimilarity) functions, fuzzy negations and aggregations are also defined in such lattice structure, proving a complete comparison between any pair of interval-valued fuzzy values.

8 WIDTH-BASED INTERVAL FUZZY ENTROPY

This section introduces the study of interval entropies generated by interval-valued fuzzy aggregations and interval-valued restricted equivalence functions w.r.t. admissible $\preceq$-order.

8.1 Width-based Interval Fuzzy Entropy: Main Concepts

In the remainder of this Section, only fuzzy sets defined on a nonempty finite referential set $U = \{u_1, u_2, \ldots, u_n\}$ will be considered. In the following we provide a definition essentially equivalent to the given in (Takáč et al., 2019, Def. 39)

Definition 8.1.1. *Let $\varepsilon \in \mathbb{U}$ such that $\underline{\varepsilon} > 0$ and $\overline{\varepsilon} < 1$ and $\leq_L$ be a partial order on $\mathbb{U}$ such that $\mathbf{0}$ and $\mathbf{1}$ are the least and greatest elements. A function $\mathbb{E}_w : \mathcal{A}_{\mathbb{U}} \rightarrow \mathbb{U}$ is called a width-based interval fuzzy entropy (ω-IvE) w.r.t. $\langle \leq_L, \varepsilon \rangle$ if it satisfies the following conditions:*

($\mathbb{E}_w 1$) $\mathbb{E}_\omega(\mathbb{A}) = \mathbf{0}$ *iff* $\mathbb{A}$ *is crisp;*

($\mathbb{E}_w 2$) $\mathbb{E}_\omega(\widetilde{\varepsilon}) = [1 - \omega(\varepsilon), 1]$*;*

($\mathbb{E}_w 3$) $\mathbb{E}_\omega(\mathbb{A}) \leq_L \mathbb{E}_\omega(\mathbb{B})$ *if for all* $u \in U$*,* $\omega(\mathbb{A}(u)) = \omega(\mathbb{B}(u))$ *and, either* $\mathbb{A}(u) \leq_L \mathbb{B}(u) \leq_L \varepsilon$ *or* $\varepsilon \leq_L \mathbb{B}(u) \leq_L \mathbb{A}(u)$*.*

8.2 Width-based Interval Fuzzy Entropy: Main Constructions

Firstly, we consider aggregation function width-based average functions, meaning that, by the action of mean aggregations, the diameter of the interval input data is preserved in the interval output data in the expression of width-based interval fuzzy entropy (ω_A-IvE) w.r.t. a partial order $\leq_L$.

In the following, let $Av : U^2 \rightarrow U$ be an average fuzzy aggregation function. Then $\widehat{Av} : \mathcal{A}_{\mathbb{U}} \rightarrow \mathcal{A}_U$ is the function defined for each $\mathbb{A} \in \mathcal{A}_{\mathbb{U}}$ and $u \in U$ as $\widehat{Av}(\mathbb{A})(u) = Av(\underline{\mathbb{A}(u)}, \overline{\mathbb{A}(u)})$.

Proposition 8.2.1. *Let $E : \mathcal{A}_U \to \mathbb{U}$ be a fuzzy entropy w.r.t. a strong fuzzy negation N, $e \in]0,1[$ the equilibrium point of N, $\preceq$ be an admissible order on $\mathbb{U}$, $Av : \mathbb{U}^2 \to \mathbb{U}$ be an idempotent averaging aggregation function and $AV_{IV} : \mathbb{U}^n \to \mathbb{U}$ be an idempotent averaging aggregation function on $\langle\mathbb{U}), \preceq\rangle$ such that $\omega(AV_{IV}(X_1, \ldots, X_n)) = k$ whenever $\omega(X_1) = \ldots = \omega(X_n) = k$. If there is $\varepsilon = [e_1, e_2] \in \mathbb{U}$ such that*

(Av1) $Av(e_1, e_2) = e$*;*

(Av2) $Av(\underline{X}, \overline{X}) \leq e$ *whenever* $X \preceq \varepsilon$.

then the function $\mathbb{E}_w : \mathcal{A}_\mathbb{U} \to \mathbb{U}$ defined by

$$\mathbb{E}_\omega(\mathbb{A}) = [E(\widehat{Av}(\mathbb{A})) \cdot (1 - \omega(AV_{IV}(\mathbb{A}(u_1), \ldots, \mathbb{A}(u_n)))), E(\widehat{Av}(\mathbb{A}))]$$

is a width-based interval fuzzy entropy w.r.t. $(\leq_L, \varepsilon)$.

Proof. Let $\mathbb{A} \in \mathcal{A}_\mathbb{U}$, the following holds

$(\mathbb{E}_{\omega 1})$ $\mathbb{E}_\omega(\mathbb{A}) = \mathbf{0} \Leftrightarrow E(\widehat{Av}(\mathbb{A})) = 0 \Leftrightarrow \widehat{Av}(\mathbb{A})$ is crisp, by(E1) $\Leftrightarrow Av(\underline{\mathbb{A}(u)}, \overline{\mathbb{A}(u)}) \in \{0,1\}$ for each $u \in U \Leftrightarrow \mathbb{A}(u) = \mathbf{0}$ or $\mathbb{A}(u) = \mathbf{1}$ for each $u \in U \Leftrightarrow \mathbb{A}$ is crisp.

$(\mathbb{E}_w 2)$ Since, for each $u \in U$, $\widetilde{\varepsilon}(u) = \varepsilon$ then $\widehat{Av}(\widetilde{\varepsilon})(u) = Av(\underline{\varepsilon}, \overline{\varepsilon}) = Av(e_1, e_2) = e$, i.e. $\widehat{Av}(\widetilde{\varepsilon}) = \tilde{e}$. So, $\mathbb{E}_\omega(\widetilde{\varepsilon}) = [E(\widehat{Av}(\widetilde{\varepsilon})) \cdot (1 - \omega(AV_{IV}(\widetilde{\varepsilon}(u_1), \ldots, \widetilde{\varepsilon}(u_n)))),\ E(\widehat{Av}(\widetilde{\varepsilon}))] = [E(\tilde{e}) \cdot (1 - \omega(AV_{IV}(\varepsilon, \ldots, \varepsilon))), E(\tilde{e})] = [1 - \omega(\varepsilon), 1]$.

$(\mathbb{E}_w 3)$ Let $\mathbb{A}, \mathbb{B} \in \mathcal{F}_{IV}(U)$ such that for each $u \in U$, $\omega(\mathbb{A}(u)) = \omega(\mathbb{B}(u))$ and $\mathbb{A}(u) \preceq \mathbb{B}(u) \preceq \varepsilon$. Since $\omega(\mathbb{A}(u)) = \omega(\mathbb{B}(u))$, $\mathbb{A}(u)$ and $\mathbb{B}(u)$ are comparable w.r.t. the product order $\leq$. So, since $\preceq$ refines the $\leq$ and by conditions **(Av1)** and **(Av2)**, we have that $Av(\underline{A(u)}, \overline{A(u)}) \leq Av(\underline{B(u)}, \overline{B(u)}) \leq Av(e_1, e_2) = e$ for each $u \in U$. Hence, $\widehat{Av}(\mathbb{A})(u) \leq \widehat{Av}(\mathbb{B})(u)$. So, $E(\widehat{Av}(\mathbb{A})) \leq E(\widehat{Av}(\mathbb{B}))$. Thus, since $1 - \omega(AV_{IV}(\mathbb{A}(u_1), \ldots, \mathbb{A}(u_n))) = 1 - \omega(AV_{IV}(\mathbb{B}(u_1), \ldots, \mathbb{B}(u_n)))$ then $[E(\widehat{Av}(\mathbb{A})) \cdot (1 - \omega(AV_{IV}(\mathbb{A}(u_1), \ldots, \mathbb{A}(u_n)))), E(\widehat{Av}(\mathbb{A}))] \leq [E(\widehat{Av}(\mathbb{B})) \cdot (1 - \omega(AV_{IV}(\mathbb{B}(u_1), \ldots, \mathbb{B}(u_n)))), E(\widehat{Av}(\mathbb{B}))]$ and, since $\preceq$ refines $\leq$, then $\mathbb{E}_\omega(\mathbb{A}) \preceq \mathbb{E}_\omega(\mathbb{B})$. Analogously, for each $u \in U$, one can proved that when $\omega(\mathbb{A}(u)) = \omega(\mathbb{B}(u))$ then $\varepsilon \preceq \mathbb{B}(u) \preceq \mathbb{A}(u)$.

□

Corollary 8.2.1. *Let $E : \mathcal{A}_U \to U$ be a fuzzy entropy w.r.t. a strong fuzzy negation N, $e \in (0,1)$ the equilibrium point of N and $Av : \mathbb{U}^k \to \mathbb{U}$ be the idempotent averaging fuzzy aggregation, given as $Av(a_1, \ldots, a_k) = \frac{1}{k}\Sigma_{i=1}^k a_i$ for $k \in \{2, n\}$. Consider $AV_{IV} : \mathbb{U}^n \to \mathbb{U}$ as the idempotent averaging interval-valued fuzzy aggregation $(\mathbb{U}, \preceq_{XY})$ defined by*

$$AV_{IV}(X_1, \ldots, X_n) = [\max(0, a - \epsilon), \min(a + \epsilon, 1)] \quad (120)$$

where $\epsilon = \frac{1}{2}Av(\omega(X_1), \ldots, \omega(X_n))$ *and* $a = Av(Av(\underline{X_1}, \overline{X_1}), \ldots, Av(\underline{X_n}, \overline{X_n}))$. *Then the function* $\mathbb{E}_\omega : \mathcal{A}_\mathbb{U} \to \mathbb{U}$ *defined by*

$$\mathbb{E}_\omega(\mathbb{A}) = \left[E(\widehat{Av}(\mathbb{A})) \cdot (1 - \omega(AV_{IV}(\mathbb{A}(u_1), \ldots, \mathbb{A}(u_n)))), E(\widehat{Av}(\mathbb{A}))\right] \tag{121}$$

is a width-based interval fuzzy entropy w.r.t. $(\preceq_{XY}, \varepsilon)$ *for* $Av(\varepsilon) = e$.

The previous results are related to the notion of width-based interval fuzzy entropy introduced in (Takáč et al., 2019, Proposition 40), which is related to a linear order $\preceq$ on $\mathbb{U}$ and to an interval-valued fuzzy negation $\mathbb{N}$ w.r.t. $\preceq$-order, in the terms of Proposition 8.2.1.

Corollary 8.2.2. *Let* $\varepsilon \in \mathbb{U}^+$, *and* $A \colon \mathbb{U} \to \mathbb{U}$ *be an admissible interleaving. Consider* $E : \mathcal{A}_\mathbb{U} \to \mathbb{U}$ *as a fuzzy entropy w.r.t. a strong fuzzy negation* N *such that* $e = A(\varepsilon)$ *is the equilibrium point. Taking* $Av : \mathbb{U}^2 \to \mathbb{U}$ *as the average aggregation function*

$$Av(a, b) = A([\min(a, b), \max(a, b)])$$

and $AV_{IV} : \mathbb{U}^n \to \mathbb{U}$ *be the left-median or right-median w.r.t.* $\preceq_A$ *admissible order. Then, for each* $\mathbb{A} \in \mathcal{A}_\mathbb{U}$, *the function* $\mathbb{E}_w : \mathcal{A}_\mathbb{U} \to \mathbb{U}$ *defined by*

$$\mathbb{E}_\omega(\mathbb{A}) = [E(\widehat{Av}(\mathbb{A})) \cdot (1 - \omega(AV_{IV}(\mathbb{A}(u_1), \ldots, \mathbb{A}(u_n)))), E(\widehat{Av}(\mathbb{A}))] \tag{122}$$

is a width-based interval fuzzy entropy w.r.t. $\langle \preceq_A, \varepsilon \rangle$.

Example 8.2.1. *Let* $U = \{u_1, u_2, u_3\}$ *and* $\mathbb{A} \in \mathcal{A}_\mathbb{U}$ *such that* $\mathbb{A}(u_i) = X_i$ *for* $X_1 = [0.1, 0.3]$, $X_2 = [0.5, 0.7]$ *and* $X_3 = [0.6, 0.8]$. *Taking the fuzzy entropy* $E : \mathcal{A}_\mathbb{U} \to [0, 1]$ *expressed as* $E(A) = \frac{1}{3}\sum_{i=1}^{3} 1 - |2A(u_i) - 1|$ *w.r.t. the* $\preceq_{XY}$*-order, we obtain the following results:*

(i) *Firstly, according with Corollary 8.2.1, we illustrate the interval entropy* $\mathbb{E}_\omega$ *w.r.t. the* $\preceq_{XY}$*-order, taking the IvF negation* $\mathbb{N}$ *given in Eq.(96) with the equilibrium point* $[\frac{1}{4}, \frac{3}{4}])$. *The operator* $\mathbb{E}_\omega$ *is constructed as follows:*
1. $\widehat{Av}(\mathbb{A})(u) = \frac{1}{2}(\underline{\mathbb{A}(u)} + \overline{\mathbb{A}(u)})$ *for each* $u \in U$. *Thereby,* $\widehat{Av}(\mathbb{A})(u_1) = 0.2$, $\widehat{Av}(\mathbb{A})(u_2) = 0.6$ *and* $\widehat{Av}(\mathbb{A})(u_3) = 0.7$.
2. $E(\widehat{Av}(\mathbb{A})) = \frac{1}{3}\sum_{i=1}^{3} 1 - |2\widehat{Av}(\mathbb{A})(u_i) - 1| = \frac{1}{3}(0.2 + 0.6 + 0.7) = 0.5$.
3. $AV_{IV}(\mathbb{A}(u_1), \mathbb{A}(u_2), \mathbb{A}(u_3)) = [\max(0, a - \epsilon), \min(1, a + \epsilon)] = [0.4, 0.6]$, *since* $\omega(AV_{IV}(X_1, X_2, X_3)$=0.2, $a = 0.5$ *and* $\epsilon = 0.1$.
4.$\mathbb{E}_\omega(\mathbb{A}) = [0.5 \cdot (1 - 0.2), 0.5] = [0.48, 0.5]$ *by Eq.(121).*

(ii) *And now, consider the interval entropy* $\mathbb{E}_\omega$ *w.r.t. the* $\preceq_{\mathbf{A}}$*-order, when* $\mathbf{A} : \mathbb{U} \to U$ *is defined in Eq(108). The interval entropy w.r.t.* $\langle \preceq_{\mathbf{A}}, [0, 0.8] \rangle$ *which is based on*

Corollary 8.2.2, can be obtained as follows:

1. $\widehat{Av}(\mathbb{A})(u) = \mathbf{A}(\underline{\mathbb{A}(u)}, \overline{\mathbb{A}(u)})$. *Then we have that* $\widehat{Av}(\mathbb{A})([0.1, 0.3]) = 0.13$*,* $\widehat{Av}(\mathbb{A})([0.5, 0.7]) = 0.57$ *and* $\widehat{Av}(\mathbb{A})([0.6, 0.8]) = 0.68$*.*

2.$E(\widehat{Av}(\mathbb{A})) = \frac{1}{3}\sum_{i=1}^{3} 1 - |2\widehat{Av}(\mathbb{A})(u_i) - 1| = \frac{1}{3}\Sigma_{i=1}^{n}(0.26, 0.86, 0.64) = 0.58\tilde{6}$*;*

3.Taking M *as the median, then* $M(\mathbf{A}[0.1, 0.3], \mathbf{A}[0.5, 0.7], \mathbf{A}[0.6, 0.8]) = M(0.13, 0.57, 0.68) = 0.57$*. Then* $\omega(AV_{IV}(\mathbb{A})) = \omega([0.5, 0.7]) = 0.2$*, since* $AV_{IV}(\mathbb{A}) = \mathbf{A}^{(-1)}M_{i=1}^{3}\mathbf{A}(u_i) = \mathbf{A}^{(-1)}(0.57) = [0.5, 0, 7]$*;*

4. $\mathbb{E}_\omega(\mathbb{A}) = [0, 58\tilde{6} \cdot (1 - 0.2), 0, 58\tilde{6}] = [0.469\tilde{3}, 0, 58\tilde{6}]$*, by Eq.(122).*

5. $Av_{IV}{}_{i=1}^{3}(X_i) = \frac{1}{3}\Sigma_{i=1}^{3}(X_i) = [0.4, 0.6]$ *and* $\omega(Av_{IV}(X_i)) = 0.2$*.*

6. $\mathbb{E}_\omega(\mathbb{A}) = [0.58 \cdot (1 - 0.2), 0.58] = [0.46, 0.58]$.

Proposition 8.2.2. *Let* $\preceq$ *be an admissible order on* $\mathbb{U}$ *which refines the* $\leq$*-order,* $\mathbb{N}$ *be a frontier interval-valued fuzzy negation on* $\mathbb{U}, \preceq)$ *with* $\varepsilon \in \mathbb{U}$ *as the equilibrium interval,* $\mathbb{S}_w$ *be a width preserving interval-valued restricted equivalence function w.r.t. the* $\preceq$*-order and* $M_{IV} : \mathbb{U}^n \to \mathbb{U}$ *be an idempotent averaging w.r.t. the* $\preceq$*-order. If* M_{IV} *satisfy* $\mathbb{M}4$ *then* $\mathbb{E}_{\mathbb{S},w} : \mathcal{A}_\mathbb{U} \to \mathbb{U}$ *defined by*

$$\mathbb{E}_{\mathbb{S},w}(\mathbb{A}) = M_{IV}(\mathbb{S}_\omega(\mathbb{A}(u_1), \mathbb{N}(\mathbb{A}(u_1))), \ldots, \mathbb{S}_\omega(\mathbb{A}(u_n), \mathbb{N}(\mathbb{A}(u_n))))$$

is a width-based interval fuzzy entropy w.r.t. the $(\preceq, \varepsilon)$*.*

Proof. For $\mathbb{A}, \mathbb{B} \in \mathcal{A}_\mathbb{U}$, the following holds:

$(\mathbb{E}_w1)$ $\mathbb{E}_{\mathbb{S},w}(\mathbb{A}) = \mathbf{0}$ iff $M_{IV}(\mathbb{S}_\omega(\mathbb{A}(u_1), \mathbb{N}(\mathbb{A}(u_1))), \ldots, \mathbb{S}_\omega(\mathbb{A}(u_n), \mathbb{N}(\mathbb{A}(u_n)))) = \mathbf{0}$ iff, by $\mathbb{M}4$, $\mathbb{S}_\omega(\mathbb{A}(u_i), \mathbb{N}(\mathbb{A}(u_i))) = \mathbf{0}$ for each $i = 1, \ldots, n$ iff, because $\mathbb{N}$ is frontier $\mathbb{A}(u_1) \in \{\mathbf{0}, \mathbf{1}\}$ iff $\mathbb{A}$ is crisp.

$(\mathbb{E}_w2)$ By Proposition , M_{IV} is idempotent, ε is equilibrium interval of $\mathbb{N}$ and based on $(\mathbb{S}2)$ property,

$$\begin{aligned}\mathbb{E}_{\mathbb{S},w}(\tilde{\varepsilon}) &= M_{IV}(\mathbb{S}_\omega(\tilde{\varepsilon}(u_1), \mathbb{N}(\tilde{\varepsilon}(u_1))), \ldots, \mathbb{S}_\omega(\tilde{\varepsilon}(u_n), \mathbb{N}(\tilde{\varepsilon}(u_n))))\\ &= M_{IV}(\mathbb{S}_\omega(\varepsilon, \mathbb{N}(\varepsilon)), \ldots, \mathbb{S}_\omega(\varepsilon, \mathbb{N}(\varepsilon))) = M_{IV}(\mathbb{S}_\omega(\varepsilon, \varepsilon), \ldots, \mathbb{S}_\omega(\varepsilon, \varepsilon))\\ &= \mathbb{S}_\omega(\varepsilon, \varepsilon) = [1 - \omega(\varepsilon), 1]\end{aligned}$$

$(\mathbb{E}_w3)$ Let $\mathbb{A}, \mathbb{B} \in \mathcal{A}_\mathbb{U}$ such that for each $u \in U$, $\omega(\mathbb{A}(u)) = \omega(\mathbb{B}(u))$ and $\mathbb{A}(u) \preceq \mathbb{B}(u) \preceq \varepsilon$. Then, $\mathbb{A}(u) \preceq \mathbb{B}(u) \preceq \varepsilon \preceq \mathbb{N}(\mathbb{B}(u)) \preceq \mathbb{N}(\mathbb{A}(u))$ for each $u \in U$. So, by $(\mathbb{S}4)$ property, $\mathbb{S}_\omega(\mathbb{A}(u), \mathbb{N}(\mathbb{A}(u))) \preceq \mathbb{S}_\omega(\mathbb{B}(u), \mathbb{N}(\mathbb{B}(u)))$ for each $u \in U$. Hence, we obtain that

$$\begin{aligned}&M_{IV}(\mathbb{S}_\omega(\mathbb{A}(u_1), \mathbb{N}(\mathbb{A}(u_1))), \ldots, \mathbb{S}_\omega(\mathbb{A}(u_n), \mathbb{N}(\mathbb{A}(u_n)))) \preceq\\ &\qquad \preceq M_{IV}(\mathbb{S}_\omega(B(u_1), \mathbb{N}(\mathbb{B}(u_1))), \ldots, \mathbb{S}_\omega(\mathbb{B}(u_n), \mathbb{N}(\mathbb{B}(u_n)))).\end{aligned}$$

So, $\mathbb{E}_{\mathbb{S},w}(\mathbb{A}) \preceq \mathbb{E}_{\mathbb{S},w}(\mathbb{B})$. And, $\omega(\mathbb{A}(u)) = \omega(\mathbb{B}(u))$ and $\varepsilon \preceq \mathbb{B}(u) \preceq \mathbb{A}(u), \forall u \in U$, is analogously proved.

□

Corollary 8.2.3. *Let $(\omega_1, \ldots, \omega_n) \in (0,1)^n$ such that $\Sigma_{i=1}^n \omega_i = 1$, $\mathbb{S}_w$ be a width preserving interval-valued restricted equivalence function w.r.t. $\preceq_{XY}$. Then $\mathbb{E}_{\mathbb{S},w} : \mathcal{A}_U \to \mathbb{U}$ defined by*

$$\mathbb{E}_{\mathbb{S},w}(\mathbb{A}) = \Sigma_{i=1}^n \omega_i \cdot \mathbb{S}_\omega(\mathbb{A}(u_i), \mathbb{N}_{XY}(\mathbb{A}(u_i))) \tag{123}$$

is a width-based interval fuzzy entropy w.r.t. $\langle \preceq_{XY}, [0.5, 0.5] \rangle$.

Proof. Straightforward. □

Example 8.2.2. *Let χ be a universe set and $\mathbb{A} \in \mathcal{A}_{\mathbb{U}}$ as defined in Ex. 8.2.1 (i). Considering the width preserving interval-valued restricted equivalence function presented in Example 7.3.5, given in Eq.(118). Then, we have that:*

$$\mathbb{S}(X_1, \mathbb{N}_{XY}(X_1)) = [1 - |0.2 - 0.8| - 0.2, 1 - |0.2 - 0.8|] = [0.2, 0.4];$$
$$\mathbb{S}(X_2, \mathbb{N}_{XY}(X_2)) = [1 - |0.6 - 0.4| - 0.2, 1 - |0.6 - 0.4|] = [0.6, 0.8];$$
$$\mathbb{S}(X_3, \mathbb{N}_{XY}(X_3)) = [1 - |0.7 - 0.3| - 0.2, 1 - |0.7 - 0.3|] = [0.4, 0.6].$$

Therefore, considering the arithmetic means, $\mathbb{E}_{\mathbb{S},w}(\mathbb{A}) = \frac{1}{3}([0.2, 0.4] + [0.6, 0.8] + [0.4, 0.6]) = [0.3, 0.6]$.

Corollary 8.2.4. *Let $AV_{IV} : \mathbb{U}^n \to \mathbb{U}$ be the left-median or right-median with respect the $\preceq_{\mathbf{A}}$ admissible order, $\varepsilon \in \mathbb{U}^+$, $\mathbb{N}_{\mathbf{A}}$ be an interval-valued fuzzy negation w.r.t. $\preceq_{\mathbf{A}}$ and equilibrium point ε, $\mathbb{S}_{\mathbf{A}}$ be the width preserving interval-valued restricted equivalence function w.r.t. $\preceq_{\mathbf{A}}$ defined in Eq.(118). Then, the function $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},w} : \mathcal{A}_{\mathbb{U}} \to \mathbb{U}$ given as*

$$\mathbb{E}_{\mathbb{S}_{\mathbf{A}},w}(\mathbb{A}) = AV_{IV}(\mathbb{S}_{\mathbf{A}}(\mathbb{A}(u_1), \mathbb{N}_{\mathbf{A}}(\mathbb{A}(u_1))), \ldots, \mathbb{S}_{\mathbf{A}}(\mathbb{A}(u_n), \mathbb{N}_{\mathbf{A}}(\mathbb{A}(u_n)))) \tag{124}$$

is a width-based interval fuzzy entropy w.r.t. $(\preceq_{\mathbf{A}}, \varepsilon)$.

Proof. Straightforward. □

Example 8.2.3. *Let $\mathbb{A} \in \mathcal{A}_{\mathbb{U}}$ as in Ex. 8.2.1 and $\mathbb{N}_e^{\mathbf{A}}$ as given in Eq.(116) when $e = 0.8$. Consider the restricted equivalence functions $\mathbb{S}_{\mathbf{A}}$ given by Eq.(118) and reported here as:*

$$\mathbb{S}_{\mathbf{A}}(X, \mathbb{N}_{\mathbf{A}}(X)) = [\min(a, 1 - \omega(X), 1 - \omega(\mathbb{N}_{\mathbf{A}}(X))), \max(a, 1 - |\mathbf{A}(X) - \mathbf{A}(\mathbb{N}_{\mathbf{A}})|)].$$

And, we obtain:

$$\begin{aligned}\mathbb{S}_{\mathbf{A}}(X_1, \mathbb{N}_{\mathbf{A}}(X_1)) &= \mathbb{S}_{\mathbf{A}}([0.1, 0.3], [0.65, 0.97]) = [\min(0.1863, 0.8, 0.32)), \max(0.1863, 0.4325)] \\ &= [0.1863, 0.4325]; \\ \mathbb{S}_{\mathbf{A}}(X_2, \mathbb{N}_{\mathbf{A}}(X_2)) &= \mathbb{S}_{\mathbf{A}}([0.5, 0.7], [0.55, 0.87]) = [\min(0.9731, 0.8, 0.68)), \max(0.9731, 0.9843)] \\ &= [0.68, 0.9843]; \\ \mathbb{S}_{\mathbf{A}}(X_3, \mathbb{N}_{\mathbf{A}}(X_3)) &= \mathbb{S}_{\mathbf{A}}([0.6, 0.8], [0.3, 0.8]) = [\min(0.5588, 0.8, 0.5)), \max(0.5588, 0.7)] \\ &= [0.5, 0.7].\end{aligned}$$

So, $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},w}(\mathbb{A}) = \min([0.1863, 0.4325], [0.68, 0.9843], [0.5, 0.7]) = [0.1863, 04325]$.

8.3 Width-based interval fuzzy entropy: Methodology

The methodology ω_A-IvE, proposed to measure the disorganized information, is applied to data obtained from the interval-valued fuzzy controller of the FuzzyNetClass.foi definido?

The procedure methods are introduced in Algorithm 1, considering the concept of width-based interval-valued fuzzy entropy presented in Definition 8.1.1. The entropy methods are generated by interval-valued aggregations, including strict interval-valued fuzzy negations with equilibrium intervals and the width-based interval-valued restricted equivalence functions, as presented by constructions in Corollary 8.2.1 and Corollary 8.2.2 related to Proposition 8.2.1. Other constructions are presented in Corollaries 8.2.1, 8.2.2, 8.2.3 and 8.2.4, which are derived from Proposition 8.2.2.

Now we illustrate the algorithm approach based on the theoretical studies related to five interval entropy measures generated by interval-valued fuzzy aggregations and interval-valued restricted equivalence functions w.r.t. the admissible $\preceq$-order.

For that, let $\mathbb{A}(u_i) = X_i \in \mathbb{U}$ be the interval-valued fuzzy value of an element $u_i \in U$ in $\mathbb{A} \in \mathcal{A}_{\mathbb{U}}$, and $E^N : U \to U$ be the fuzzy normal-entropy given as $E^N(X_i) = 1 - |2X_i - 1|$. In addition, to compare our proposal we applied the entropy on (Takáč et al., 2019).

Method 1 Based on Corollary 8.2.1, this method is defined by Eq.(121), by taking the interval-valued average function as the arithmetic mean. Thus, the interval-valued entropy $\mathbb{E}_\omega : \mathcal{A}_{\mathbb{U}} \to \mathbb{U}$ related to $(\mathbb{U}, \preceq_{XY})$ can be expressed as follows:

$$\mathbb{E}_\omega(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\omega^N(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}_\omega^N(X_i),$$

Algorithm 1: Methods to Analyse Video Streaming Traffic Information via Interval Entropy Measure

Input: IvFS quantifying each language variable

Output: ω_A-IvE measuring the information related to each IvFS

1. Select the Interval Entropy Method, Admissible $\preceq$-order and Aggregation
2. Select the dataset.
3. Select the Attribute and the IvFS based on Att_LV_List

for $IvFS$ ***in*** $Att\_LV - List$ **do**
  $\mathbb{E}_n \leftarrow$ GETINTERVALENTROPY$(\mathbb{A}_n)$
  for n ***in*** $EntropyList$ **do**
    1.1 Apply the selected method to each membership degree for all elements in the IvFS selected;
    1.2 Aggregate Final Results in $Entropy - List$;
  end
  2. Present $Entropy - List\_Result$;
end
4. Apply the $\preceq$-order to compare results from $Entropy - List\_Result$

whenever $\mathbb{E}^N_\omega : \mathbb{U} \to \mathbb{U}$ is given by the following expression

$$\mathbb{E}^N_\omega(X_i) = \begin{cases} \left[(\overline{X_i} + \underline{X_i}) \cdot \mathcal{K}, (\overline{X_i} + \underline{X_i})\right], \text{ if } \overline{X_i} + \underline{X_i} \leq 1 \\ \\ \left[2 - (\overline{X_i} + \underline{X_i}) \cdot \mathcal{K}, 2 - (\overline{X_i} + \underline{X_i})\right], \text{ otherwise,} \end{cases} \tag{125}$$

and taking $\mathcal{K} = (1 - \omega(AV_{IV}(X_1), \ldots, X_n)))$.

Method 2 Based on Corollary 8.2.2, this method is defined by Eq.(122), taking the average fuzzy as the arithmetic means. Thus, for all $u_i \in U$, let $(\mathbb{A}(u_i) = X_i \in \mathbb{U}$, the interval-valued entropy $\mathbb{E}_{\mathbf{A}} : \mathcal{A}_{\mathbb{U}} \to \mathbb{U}$ related to $(\mathbb{U}, \preceq_{XY})$ is given as follows:

$$\mathbb{E}_{\mathbf{A}}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^N_{\mathbf{A}}(X_i).$$

whenever $\mathbb{E}^N_{\mathbf{A}} : \mathbb{U} \to \mathbb{U}$ is obtained as

$$\mathbb{E}^N_{\mathbf{A}}(X_i) = \begin{cases} \left[2\mathbf{A}(\overline{X_i}, \underline{X_i}) \cdot \mathcal{K}_{\mathbf{A}}, 2\mathbf{A}(\overline{X_i}, \underline{X_i})\right], \text{ if } 2\mathbf{A}(X_i) \leq 1 \\ \\ \left[2 - 2\mathbf{A}(\overline{X_i}, \underline{X_i}) \cdot \mathcal{K}_{\mathbf{A}}, 2 - 2\mathbf{A}(\overline{X_i}, \underline{X_i})\right], \text{ otherwise,} \end{cases} \tag{126}$$

and taking $\mathcal{K}_{\mathbf{A}} = 1 - \omega(M^n_{i=1}\mathbf{A}(X_i))$, when M is the median aggregation function.

Method 3 Based on Corollary 8.2.3, this method is defined by Eq.(123), taking the average fuzzy as the arithmetic mean. Thus, the interval-valued entropy $\mathbb{E}_{\mathbb{S},\omega}$:

$\mathcal{A}_{\mathbb{U}} \to \mathbb{U}$ is given as follows:

$$\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{S},\omega}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{S},\omega}(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{S}_{\omega}(X_i, \mathbb{N}(X_i)),$$

whenever we take $\mathbb{S}_{\omega} : \mathbb{U}^2 \to \mathbb{U}$ as the width-based interval-valued restricted equivalence function w.r.t. the $\preceq_{XY}$-order, $\mathbb{N}$ as the strong interval-valued fuzzy negation w.r.t. the $\preceq_{XY}$-order given in Eq.(96) and $\mathbb{M}$ as the medium point of an interval. Thus, $\mathbb{E}_{\mathbb{S},\omega}$ is defined as follows:

$$\mathbb{S}_{\omega}(X_i, \mathbb{N}_{XY}(X_i)) = \left[1 - \alpha - \frac{1}{2}(\omega(X_i) + \omega(\mathbb{N}_{XY}(X_i)), 1 - \alpha\right]$$

where $\alpha = |\mathbb{M}(X_i) - \mathbb{M}(\mathbb{N}_{XY}(X_i))|$. And then, $\mathbb{S}_{\omega}$ can also be expressed as presented in the following:

$$\mathbb{S}_{\omega}(X_i, \mathbb{N}_{XY}(X_i)) = \begin{cases} \left[\frac{1}{2}(\overline{X_i} + \underline{X_i}), \overline{X_i} + \underline{X_i}\right] \text{ if } \overline{X_i} + \underline{X_i} \leq 1; \\ \left[1 - \frac{1}{2}(\overline{X_i} + \underline{X_i}), 2 - (\overline{X_i} + \underline{X_i})\right], \text{ otherwise.} \end{cases} \tag{127}$$

Method 4 Based on Corollary 8.2.4, this method is defined by Eq.(124), taking the average fuzzy as the arithmetic mean. Thus, the interval-valued entropy $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega} : \mathcal{A}_{\mathbb{U}} \to \mathbb{U}$ is given as follows:

$$\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{S}_{\mathbf{A}},\omega}(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{S}_{\mathbf{A}}(X_i, \mathbb{N}_{\mathbf{A}}(X_i)),$$

whenever we take $\mathbb{S}_{\mathbf{A}} : \mathbb{U}^2 \to \mathbb{U}$ as the width-based interval-valued restricted equivalence function w.r.t. the $\preceq_{\mathbf{A}}$-order, expressed here as follows

$$\mathbb{S}_{\mathbf{A}}(X_i, \mathbb{N}_{\mathbf{A}}(X_i)) = \tag{128}$$

$$\begin{cases} [\min(\mathcal{K}, 1-\omega(X_i), 1-\omega(\mathbb{N}_{\mathbf{A}}(X_i))), \max(\mathcal{K}, 1-\mathbf{A}(\mathbb{N}_{\mathbf{A}}(X_i))+\mathbf{A}(X_i))], \\ \text{if } \mathbf{A}(X_i) \leq \mathbf{A}(\mathbb{N}_{\mathbf{A}}(X_i)); \\ [\min(\mathcal{K}^{-1}, 1 - \omega(X_i), 1 - \omega(\mathbb{N}_{\mathbf{A}}(X_i))), \max(\mathcal{K}^{-1}, 1 - \mathbf{A}(X_i) + \mathbf{A}(\mathbb{N}_{\mathbf{A}}(X_i)))], \\ \text{otherwise.} \end{cases}$$

by taking $\mathbf{A}$ as the aggregation given in Eq.(105) and $\mathcal{K}_{\mathbf{A}} = \frac{\mathbf{A}(X_i)}{\mathbf{A}(\mathbb{N}_{\mathbf{A}}(X_i))}$.

Method 5 Again, by Corollary 8.2.4, the next method is defined by Eq.(124), taking the average fuzzy as the arithmetic mean. Thus, the interval-valued entropy $\mathbb{E}_{\mathbb{R},\omega}$:

$\mathcal{A}_{\mathbb{U}} \rightarrow \mathbb{U}$ is given as follows:

$$\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{R},\omega}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}_{\mathbb{R},\omega}(X_i) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{R}_{\omega}(X_i, \mathbb{N}_e(X_i)),$$

whenever we take $\mathbb{R}_{\mathbf{A}} : \mathbb{U}^2 \rightarrow \mathbb{U}$ as the width-based interval-valued restricted equivalence function w.r.t. the $\preceq_{\mathbf{A}}$-order, as given in Eq.(119), and expressed as

$$\mathbb{R}_{\mathbf{A}}(X, \mathbb{N}_e(X)) = \\ [\max(0, N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(\mathbb{N}_e(X))|) - \max(\omega(X), \omega(\mathbb{N}_e(X)))), N_{\mathrm{e}}(|\mathbf{A}(X) - \mathbf{A}(\mathbb{N}_e(X))|)] \quad (129)$$

Method 6 In order to compare the above proposed methods we also consider the width-based interval fuzzy entropy related to $(\mathbb{U}, \preceq_{XY})$ introduced in (Takáč et al., 2019, Example 30), and reported below:

$$\mathbb{E}^{p}_{IV}(\mathbb{A}) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N}(\mathbb{A}(u_i)) = \frac{1}{n}\sum_{i=1}^{n} \mathbb{E}^{N_{XY}}(X_i),$$

when $\mathbb{E}^{p}_{IV} : \mathbb{U} \rightarrow \mathbb{U}$ is expressed as follows:

$$\mathbb{E}^{p}_{IV}(X_i) = \left[1 - |\underline{X}_i + \overline{X}_i - 1| - (\overline{X}_i - \overline{X}_i), 1 - |\underline{X}_i + |\overline{X}_i - 1|\right]$$

Then, new results comparing $\mathbb{E}^{p}_{IV}$, from Method 01 to Method 05 are discussed in the next section.

8.4 Summary

This chapter presents the main contribution of this research work, introducing the concepts and related constructions of a new methodology based on ω-preserving fuzzy connectives, under which we are able to propose new methods for obtaining the interval-valued fuzzy entropy.

The methodology is generated by ω-preserving interval-valued fuzzy aggregations and interval-valued fuzzy restricted equivalence function w.r.t. the admissible orders.

In particular, the well-known Xu and Yager's $\preceq_{XY}$-order is explored in order to express the list of five proposed methods. We also should highlight the use of DDI functions defining the $\preceq_{\mathbf{A}}$-order applied in the expression of two interval-valued entropy constructive methods.

The algebraic expression defining each one of the five methods is presented, including illustrative examples for a more intuitive comprehension, instigating their application in the next chapters.

9 INTERVAL-VALUED INTUITIONISTIC FUZZY ENTROPY

The concept of entropy measuring the fuzziness of a fuzzy set was introduced by De Luca and Termini (Luca; Termini, 1972) in order to measure how far a fuzzy set (FS) is from a crisp one. Since then, this concept has been adapted to the different extensions of FS and with different interpretations, as in modeling type-2 fuzzy sets (DE MIGUEL et al., 2017) (Xu; Shen, 2014), interpreting vague sets (Zhang; Jiang, 2008b), dealing with intuitionistic fuzzy set (Wei; Gao; Guo, 2012), (Ye, 2010),(Verma; Sharma, 2013), (Liu; Ren, 2014a) and also modeling interval-valued intuitionistic fuzzy sets (Jing; Min, 2013) (Zhang; Jiang, 2008b), all of them measure how far the considered extension is from a fuzzy set of reference.

In this sense, it is worth mentioning the following concepts: the Atanassov intuitionistic fuzzy entropy measure, given by Szmidt and Kacprzyk (Szmidt; Kacprzyk, 2001) to measure how far A-IFS is from a crisp set. The entropy for interval-valued fuzzy sets (IvFSs) is defined by Burillo and Bustince (Burillo; Bustince, 1996), which measures how far an IvFS or A-IFS is from an FS.

The generalized interval-valued intuitionistic fuzzy index seems to be suitable to deal with measures of entropy in A-IvIFS, modeling uncertainty, and imprecision in membership and non-membership functions.

Following this approach, this chapter generalizes results from (Bustince et al., 2011), discussing properties related to Atanassov's interval-valued intuitionistic fuzzy entropy (A-IvIFE) which are obtained by the action of an interval-valued aggregation applied to the generalized interval-valued intuitionistic fuzzy index, using admissible orders on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$.

9.1 Interval-valued Intuitionistic Fuzzy Indexes on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$

This section presents a proposal of obtain IvIFE based on the aggregation of A-GIvIFIx, considering this operator defined on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$.

Thus, this section contemplates the study of relevant admissible orders on $\tilde{\mathbb{U}}$, demanding a review reach from interval-valued intuitionistic fuzzy indexes, referred to as an interval-valued intuitionistic extension of the score, the accuracy, and the hesitancy indexes.

The main results presented in (Lee, 2009) and (Ze-shui, 2007) extend the score measure from $\tilde{U}$ to $\tilde{\mathbb{U}}$. So, the score function $S : \tilde{\mathbb{U}} \to [-1, 1]$ is defined as follows:

$$\mathbb{S}(X, Y) = \frac{1}{2}(\overline{X} + \underline{X} - (\overline{Y} + \underline{Y})), \forall (X, Y) \in \tilde{\mathbb{U}}. \tag{130}$$

Consider the preorder in $\tilde{\mathbb{U}}$ given as $(X_1, Y_1) \leq_{\tilde{\mathbb{U}}} (X_2, Y_2) \Leftrightarrow \mathbb{S}((X_1, Y_1)) \leq \mathbb{S}(X_2, Y_2)$ and the equivalence relation given as $(X_1, Y_1) \equiv_{\tilde{\mathbb{U}}} (X_2, Y_2) \Leftrightarrow \mathbb{S}((X_1, Y_1)) = \mathbb{S}((X_2, Y_2))$. The binary relation is given as

$$(X_1, Y_1) <_{\mathbb{S}} (X_2, Y_2) \Leftrightarrow \mathbb{S}(X_1, Y_1) < \mathbb{S}(X_2, Y_2), \text{ and}$$
$$(X_1, Y_1) \equiv_{\mathbb{S}} (X_2, Y_2) \Leftrightarrow \mathbb{S}(X_1, Y_1) = \mathbb{S}(X_1, Y_1),$$

is a partial order on $\langle \tilde{\mathbb{U}}, \leq_{\tilde{\mathbb{U}}} \rangle$.

Additionally, based on (Ze-shui, 2007), by taking $\omega(X)) = \overline{X} - \underline{X}$, the accuracy function $\mathbb{H} : \tilde{\mathbb{U}} \to [0, 1]$ and two other functions $\mathbb{T} : \tilde{\mathbb{U}} \to [-1, 1]$ and $\mathbb{G} : \tilde{\mathbb{U}} \to [0, 1]$ can also be defined as follows

$$\mathbb{H}(X, Y) = \frac{1}{2}(\overline{X} + \underline{X} + \overline{Y} + \underline{Y}). \tag{131}$$
$$\mathbb{T}(X, Y) = \omega(X) - \omega(Y); \tag{132}$$
$$\mathbb{G}(X, Y) = \omega(X) + \omega(Y), \tag{133}$$

These function are applied to define a total order in $\tilde{\mathbb{U}}$.

9.1.1 Admissible Orders on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$

In order to compare A-IvIFS, the Xu and Yager's admissible $\preceq^*_{XY}$-order is considered in this section.

Theorem 9.1.1. *Let $\mathbb{A}_I, \mathbb{B}_I \in \mathcal{A}_{\tilde{\mathbb{U}}}$. Then, we have that $\mathbb{A}_I \preceq^*_{XY} \mathbb{B}_I$ if, and only if, $\forall \mathrm{x} \in \chi$, the following inequality holds:*

$$\mathbb{A}_I \prec^*_{XY} \mathbb{B}_I \Leftrightarrow$$
$$\Leftrightarrow \begin{cases} \mathbb{S}_{\mathbb{A}_I}(\mu_{\mathbb{A}_I}(\mathrm{x}), (\nu_{\mathbb{A}_I}(\mathrm{x})) \leq \mathbb{S}_{\mathbb{B}_I}(\mu_{\mathbb{B}_I}(\mathrm{x}), (\nu_{\mathbb{B}_I}(\mathrm{x})) \text{ or} \\ \mathbb{S}_{\mathbb{A}_I}(\mu_{\mathbb{A}_I}(\mathrm{x}), (\nu_{\mathbb{A}_I}(\mathrm{x})) = \mathbb{S}_{\mathbb{B}_I}(\mu_{\mathbb{B}_I}(\mathrm{x}), (\nu_{\mathbb{B}_I}(\mathrm{x})); \mathbb{H}_{\mathbb{A}_I}(\mu_{A_I}(\mathrm{x}), (\nu_{\mathbb{A}_I}(\mathrm{x})) \leq \mathbb{H}_{\mathbb{A}_I}(\mu_{\mathbb{B}_I}(\mathrm{x}), (\nu_{\mathbb{B}_I}(\mathrm{x})) \end{cases}$$

And, the corresponding equality is defined as follows:

$$A_I =_{XY} B_I \Leftrightarrow \begin{cases} \mathbb{S}_{A_I}(x,y) = \mathbb{S}_{A_I}(\mu_{A_I}(\mathrm{x}), (\nu_{A_I}(\mathrm{x})) \text{ and} \\ \mathbb{H}_{A_I}(\mu_{A_I}(\mathrm{x}), (\nu_{A_I}(\mathrm{x})) = \mathbb{H}_{A_I}(\mu_{B_I}(\mathrm{x}), (\nu_{B_I}(\mathrm{x})). \end{cases}$$

formalizing the admissible $\preceq^*_{XY}$-order w.r.t. the usual partial $\leq_{\tilde{\mathbb{U}}}$-order on $\tilde{\mathbb{U}}$.

Theorem 9.1.2. *(Wang; Li; Wang, 2009) The $\preceq^*_{WLW}$-relation defined, for each pair $((X_1,Y_1),(X_1,Y_1)) \in \tilde{\mathbb{U}}^2$, by the following expression:*

$$(X_1,Y_1) \preceq^*_{WLW} (X_2,Y_2) \Leftrightarrow \tag{134}$$
$$\begin{cases} (X_1,Y_1) <_{\mathbb{S}} (X_2,Y_2) \text{ or} \\ (X_1,Y_1) \equiv_{\mathbb{S}} (X_2,Y_2); H(X_1,Y_1) < H(X_2,Y_2) \text{ or} \\ (X_1,Y_1) \equiv_{\mathbb{S}} (X_2,Y_2); H(X_1,Y_1) = H(X_2,Y_2); T(X_1,Y_1) < T((X_2,Y_2)) \text{ or} \\ (X_1,Y_1) \equiv_{\mathbb{S}} (X_2,Y_2); H(X) = H(Y) \text{ and } T(X_1,Y_1) = T((X_2,Y_2)); G(X_1,Y_1) < G(X_2,Y_2). \end{cases}$$

is an admissible $\preceq_{\tilde{\mathbb{U}}}$-order w.r.t. the usual partial $\leq_{\tilde{\mathbb{U}}}$-order on $\tilde{\mathbb{U}}$.

In (DA SILVA; Bedregal; Santiago, 2016), the authors proposed a parametric family composed of total orders on $\tilde{\mathbb{U}}$, which is reported below.

Theorem 9.1.3. *(DA SILVA; Bedregal; Santiago, 2016) Let $\preceq_{\mathbb{U}}$-relation be a total order on $\mathbb{U}$. The $\preceq_{\tilde{\mathbb{U}}}$-relation defined, for each pair $((X_1,Y_1),(X_1,Y_1)) \in \tilde{\mathbb{U}}^2$, by the expression:*

$$(X_1,Y_1) \preceq_{\tilde{\mathbb{U}}} (X_2,Y_2) \Leftrightarrow X_1 \preceq_{\mathbb{U}} X_2 \text{ or } (X_1 = X_2 \text{ and } Y_2 \preceq_{\mathbb{U}} Y_1) \tag{135}$$

is a total order.

Since the $\preceq_{XY}$ is an admissible order on $\mathbb{U}$, the following holds from Proposition 9.1.3.

Proposition 9.1.1. *(DA SILVA; Bedregal; Santiago, 2016, Theorem 4.2) Let $\preceq_{XY}$-relation be the Xu and Yager's admissible linear order on $\mathbb{U}$. Then the binary relation $\preceq^*_{XY}$ on $\tilde{\mathbb{U}}^2$, defined, for each pair $((X_1,Y_1),(X_2,Y_2)) \in \tilde{\mathbb{U}}^2$, by the expression*

$$(X_1,Y_1) \preceq^*_{XY} (X_2,Y_2) \Leftrightarrow X_1 \preceq_{XY} X_2 \text{ or } (X_1 = X_2 \text{ and } Y_2 \preceq_{XY} Y_1) \tag{136}$$

is a (total) admissible order on $\tilde{\mathbb{U}}$.

Thus, according with (DA SILVA; Bedregal; Santiago, 2016) the ordered structured sets $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$, $\langle \tilde{\mathbb{U}}, \preceq^*_{XY} \rangle$ are bounded lattices.

9.1.2 Interval Extension of an A-GIFIx on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$

In this section, the axiomatic definition of an A-GIFIx given in (Bustince et al., 2011, Definiton 1) is extended to Atanassov's interval-valued intuitionistic approach, considering Xu and Yager admissible $\preceq_{XY}$-order.

Definition 9.1.1. *A function $\tilde{\Pi} : \tilde{\mathbb{U}} \to \mathbb{U}$ is called a generalized interval-valued intuitionistic fuzzy index associated with a strong IvFN $\mathbb{N}$ (A-GIvIFIx($\mathbb{N}$)) w.r.t. a $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$ if, for all $\tilde{X} = (X_1, X_2), \tilde{Y} = (Y_1, Y_2) \in \tilde{\mathbb{U}}$, it holds that:*

$\tilde{\Pi}1$: $\tilde{\Pi}(X_1, X_2) = \mathbf{1} \Leftrightarrow X_1 = X_2 = \mathbf{0}$;

$\tilde{\Pi}2$: $\tilde{\Pi}(X_1, X_2) = \mathbf{0} \Leftrightarrow X_1 + X_2 = \mathbf{1}$;

$\tilde{\Pi}3$: $(Y_1, Y_2) \preccurlyeq_{\tilde{\mathbb{U}}} (X_1, X_2) \Rightarrow \tilde{\Pi}(X_1, X_2) \preceq_{\mathbb{U}} \tilde{\Pi}(Y_1, Y_2)$;

$\tilde{\Pi}4$: $\tilde{\Pi}(X_1, X_2) = \tilde{\Pi}(\mathbb{N}_{\mathbb{S}_I}(X_1, X_2))$ *taking* $\mathbb{N}_{S_I}$ *in Eq.(23).*

Based on Definiton 9.1.1, a class of A-GIvIFIx obtained from IvIFI and IvFN w.r.t. Xu-Yager's order $\preceq_{\mathbb{U}}$ is formalized in the next proposition.

Proposition 9.1.2. *Let $\mathbb{I}_{\mathbb{M}_\beta, \mathbb{N}} : \mathbb{U}^2 \to \mathbb{U}$ be an IvFI w.r.t. a total order given in Proposition 7.1.6, by Eq.(100). The function $\tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_\beta}, \mathbb{N}_{XY}} : \tilde{\mathbb{U}}^2 \to \tilde{\mathbb{U}}$ given as follows:*

$$\tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_\beta}, \mathbb{N}_{\mathbb{XY}}}(X_1, X_2) = \begin{cases} \mathbf{0}, \ \textit{if}\ \mathbb{N}(X_2) \preceq_{XY} X_1; \\ \mathbf{1}, \ \textit{if}\ X_2 = \mathbf{0} \textit{ and } X_1 = \mathbf{0}; \\ \left[1 - \beta \frac{3\underline{X}_2 + \overline{X}_2}{2} + (1-\beta)\frac{3\underline{X}_1 + \overline{X}_1}{2}, (1-\beta)\frac{\overline{X}_2 - \underline{X}_2}{2} - (1-\beta)\frac{\overline{X}_1 - \underline{X}_1}{2}\right], \\ \quad \textit{if}\ (1-\beta)\overline{X}_1 + \underline{X}_1 < 1 - \beta(\overline{X}_2 + \underline{X}_2) \textit{ and } X_2 \neq \mathbf{0}, X_2 \neq \mathbf{0}; \\ \left[\beta \frac{\overline{X}_2 - \underline{X}_2}{2} + (1-\beta)\frac{\overline{X}_1 - \underline{X}_1}{2}, 2 - \beta(\frac{3\overline{X}_2 - \underline{X}_2}{2}) + (1-\beta)(\frac{3\overline{X}_1 - \underline{X}_1}{2})\right], \textit{otherwise}. \end{cases} \quad (137)$$

is an A-GIvIFIx obtained from IvIFI and IvFN w.r.t. the $\preceq_{XY}$-order.

Proof. It follows from results of Propositions 7.1.6 and 7.1.5, also including Eq.(99) in Prop. 7.1.4 and taking $\mathbb{M}_\beta$ and $\mathbb{N}$ given by Eqs.(97) and (95), respectively. □

Thus, the above proposition presents a method to obtain an interval extension of the A-GIFIx, which is compatible with comparisons, enabling the ranking of results in fuzzy systems based on A-IvIFS.

Concluding, we present a member of the above class, an IvFI obtained by composition of the IvA $\mathbb{M}_{\frac{1}{2}}$ and the strong IvFN $\mathbb{N}$, given in Eqs.(98) and (96), respectively.

Example 9.1.1. *In the conditions of Proposition 7.1.5, taking $\beta = \frac{1}{2}$ and $\mathbb{N}_{XY}$ given in Eq.(96), then the function $\tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_\beta,\mathbb{N}}} : \tilde{\mathbb{U}}^2 \to \mathbb{U}$ given as follows:*

$$\tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_\beta,\mathbb{N}_{XY}}}(X_1, X_2) = \begin{cases} \mathbf{0}, \text{ if } \mathbb{N}(X_2) \preceq X_1; \\ \mathbf{1}, \text{ if } X_2 = \mathbf{0} \text{ or } X_1 = \mathbf{0}; \\ \left[1 - \frac{3\underline{X}_2 + \overline{X}_2}{4} - \frac{3\underline{X}_1 + \overline{X}_1}{4}, 1 - \frac{\overline{X}_2 - \underline{X}_2}{4} - \frac{\overline{X}_1 - \underline{X}_1}{4}\right], \\ \quad \text{if } \overline{X}_1 + \underline{X}_1 < 2 - (\overline{X}_2 + \underline{X}_2) \text{ and } X_2 \neq \mathbf{0}, X_2 \neq \mathbf{0}; \\ \left[\frac{\overline{X}_2 - \underline{X}_2}{4} + \frac{\overline{X}_1 - \underline{X}_1}{4}, 2 - (\frac{3\overline{X}_2 + \underline{X}_2}{4}) + (\frac{3\overline{X}_1 - \underline{X}_1}{4})\right], \text{ otherwise.} \end{cases}$$

is an A-GIvIFIx w.r.t. Xu-Yager's order on $\langle \tilde{\mathbb{U}}, \preceq_{XY} \rangle$

9.2 Interval Extension of Entropy on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$

Definition 9.2.1. *An interval-valued function $\mathbb{E}_I : \mathcal{A}_{\tilde{\mathbb{U}}} \to \mathbb{U}$ is called an A-IvIFE on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$ if and only if $\mathbb{E}_I$ verifies the following properties:*

$\mathbb{E}_I 1$: $\mathbb{E}_I(A_I) = \mathbf{0} \Leftrightarrow \mathbb{A}_I \in \mathcal{A}_{\tilde{\mathbb{U}}}$;

$\mathbb{E}_I 2$: $\mathbb{E}_I(\mathbb{A}_I) = \mathbf{1} \Leftrightarrow \mu_{\mathbb{A}_I}(x) = \nu_{\mathbb{A}_I}(x) = \mathbf{0}$, $\forall x \in \chi$;

$\mathbb{E}_I 3$: $\mathbb{E}_I(\mathbb{A}_I) = \mathbb{E}(\mathbb{A}_{I_C})$;

$\mathbb{E}_I 4$: *If* $\mathbb{A}_I \preceq_{\tilde{\mathbb{U}}} \mathbb{B}_I$ *then* $\mathbb{E}_I(\mathbb{A}_I) \preceq_{\mathbb{U}} \mathbb{E}_I(\mathbb{B}_I)$, $\forall \mathbb{A}_I, \mathbb{B}_I \in \mathcal{A}_{\tilde{\mathbb{U}}}$.

Considering the axiomatic Definition 9.2.1 of Atanassov Interval-valued intuitionistic fuzzy entropy, the following Theorem presents the construction of the A-IvIFE obtained through A-GIvIFIx and idempotents aggregators that satisfies the axioms of the above definition.

Theorem 9.2.1. *Consider $\chi = \{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$. Let $\mathbb{M} : \mathbb{U}^n \to \mathbb{U}$ be an IvA , $\mathbb{N}$ be a strong IvFN and $\tilde{\Pi}$ be an A-GIvIFIx($\mathbb{N}$). A function $\mathbb{E}_I : \mathcal{A}_{\tilde{\mathbb{U}}} \to \mathbb{U}$ given by*

$$\mathbb{E}_I(\mathbb{A}_I) = \mathbb{M}_{i=1}^n \tilde{\Pi}_I(\mathbb{A}_I(\mathrm{x}_i)) = \mathbb{M}_{i=1}^n \tilde{\Pi}_{\mathbb{I}}(\mu_{\mathbb{A}_I}(\mathrm{x}_i), \nu_{\mathbb{A}_I}(\mathrm{x}_i)), \forall \mathrm{x}_i \in \chi, \tag{138}$$

is an A-IvIFE in the sense of Definition 9.2.1.

Proof. Let $\mathbb{A}_{I_C}$ be the complement of $\mathbb{A}_I$ given by Eq.(74). For all $x_i \in \chi$ and $\mathbb{A}_I, \mathbb{B}_I \in \mathcal{A}_{\tilde{\mathbb{U}}}$, we have that:

$\mathbb{E}_I 1$: $\mathbb{E}_I(\mathbb{A}_I) = \mathbf{0} \Leftrightarrow \mathbb{M}_{i=1}^n \tilde{\Pi}(A_I(x_i)) = \tilde{\mathbf{0}}$. By $\mathbb{M}_I 1$, $\mathbb{E}_I(A_I) = \mathbf{0} \Leftrightarrow \mu_{\mathbb{A}_I}(\mathrm{x}_i) + \nu_{\mathbb{A}_I}(\mathrm{x}_i) = \tilde{\mathbf{1}}$. Then, by $\tilde{\Pi}2$, $\mathbb{E}_I(\mathbb{A}_I) = \tilde{\mathbf{0}} \Leftrightarrow \mathbb{A}_I \in \mathcal{A}_{\tilde{\mathbb{U}}}$.

$\mathbb{E}2$: Analogous to $\mathbb{E}_I 1$.

$\mathbb{E}3$: $\mathbb{E}_I(\mathbb{A}_{I_C}) = \mathbb{M}_{i=1}^n \tilde{\Pi}(\mathbb{A}_{I_C}(\mathrm{x}_i)) = \mathbb{M}_{i=1}^n \tilde{\Pi}(\mathbb{N}_{S_I}(X_1, X_2)) = \mathbb{M}_{i=1}^n \tilde{\Pi}(X_1, X_2)$, by $\tilde{\Pi}4$. So, the following holds $\mathbb{E}_I(\mathbb{A}_{I_C}) = \mathbb{M}_{i=1}^n \tilde{\Pi}(X_1, X_2)$. Concluding, $\mathbb{E}_I(\mathbb{A}_{I_C}) = \mathbb{E}_I(\mathbb{A}_I)$.

$\mathbb{E}4$: If $\mathbb{A}_I \preceq_{\tilde{\mathbb{U}}} \mathbb{B}_I$ then $\mathbb{A}_I(\mathrm{x}_i) \preceq_{\tilde{\mathbb{U}}} \mathbb{B}_I(\mathrm{x}_i)$, $\forall \mathrm{x}_i \in \chi$. Based on $\tilde{\Pi}3$, it holds that $\tilde{\Pi}(\mathbb{B}_I(\mathrm{x}_i)) \preceq_{\mathbb{U}} \tilde{\Pi}(\mathbb{A}_I(\mathrm{x}_i))$. By $\mathbb{M}3$, we obtain that $\mathbb{M}_{I}{}_{i=1}^n \tilde{\Pi}(\mathbb{B}_I(\mathrm{x}_i)) \preceq_{\mathbb{U}} \mathbb{M}_{i=1}^n \tilde{\Pi}(\mathbb{B}_I(x_i))$. As conclusion, we have that $\mathbb{E}_I(\mathbb{A}_I) \preceq_{\mathbb{U}} \mathbb{E}_I(\mathbb{B}_I)$.

Therefore, Theorem 9.2.1 is verified. □

The next proposition formalizes the interval extension of the constructive method to obtain interval fuzzy entropy generalized interval-valued fuzzy intuitionistic fuzzy index. Such a method extends the main results presented in (Bustince et al., 2011).

Proposition 9.2.1. *Consider $\chi = \{\mathrm{x}_1, \ldots, \mathrm{x}_n\}$. Let $\mathbb{M} : \mathbb{U}^n \to \mathbb{U}$ be an interval-valued aggregation on $\langle \tilde{\mathbb{U}}, \preceq_{XY} \rangle$. Let $\mathbb{N}_{XY}$ be a strong negation defined in Eq.(95) and $\mathbb{I}_{\mathbb{M}_\beta, \mathbb{N}} : \mathbb{U}^2 \to \mathbb{U}$ be an IvFI given by Eq.(100) which is considered in definition of the A-GIvIFIx($\mathbb{N}_{XY}$) $\tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_\beta, \mathbb{N}_{XY}}} : \tilde{\mathbb{U}}^2 \to \mathbb{U}$. Then, the operator $\mathbb{E}_I : \mathcal{A}_{\tilde{\mathbb{U}}} \to \mathbb{U}$ defined as follows*

$$\mathbb{E}_{I \tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_\beta, \mathbb{N}_{XY}}}}(\mathbb{A}_I) = \mathbb{M}_{i=1}^n \tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_\beta, \mathbb{N}_{XY}}}(\mu_{\mathbb{A}_I}(\mathrm{x}_i), \nu_{\mathbb{A}_I}(\mathrm{x}_i)), \forall \mathrm{x}_i \in \chi. \tag{139}$$

is an A-IvIFE on $\langle \tilde{\mathbb{U}}, \preceq_{XY} \rangle$.

Proof. Straightforward Proposition 9.1.2 and Proposition 9.2.1. □

Corollary 9.2.1. *In Proposition 9.2.1, consider the aggregation operator $\mathbb{M}$ as the arithmetic, and $\beta = \frac{1}{2}$ in the IvIFI $\mathbb{I}_{\mathbb{M}_\beta, \mathbb{N}_{XY}}$. For all $\mathrm{x}_i \in \chi$ and $\mathbb{A}_I \in \mathcal{A}_{\tilde{\mathbb{U}}}$, we have the following expression for an A-IvIFE*

$$\mathbb{E}_{I \tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_{\frac{1}{2}}, \mathbb{N}_{XY}}}}(\mathbb{A}_I) = \frac{1}{n} \sum_{i=1}^n \tilde{\Pi}_{\mathbb{I}_{\mathbb{M}_{\frac{1}{2}}, \mathbb{N}_{XY}}}(\mu_{\mathbb{A}_I}(\mathrm{x}_i), \nu_{\mathbb{A}_I}(\mathrm{x}_i)), \forall \mathrm{x}_i \in \chi.$$

Proof. Straightforward Proposition 9.2.1. □

9.3 Summary

The chapter presented the constructions referring to interval-valued intuitionistic fuzzy entropy treated at intervals through the aggregation of the generalized interval-valued intuitionistic fuzzy index. For these constructions, the concepts of total orders were considered, and the concepts of interval extension of the A-GIFIx and of the intuitionistic fuzzy entropy are discussed over the ordered structure $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$.

Moreover, in order to generate a new method to obtain interval-valued intuitionistic fuzzy entropy, we restricted our focus to Xu and Yager's admissible order, providing related interval extension of fuzzy implications and fuzzy negations on $\langle \tilde{\mathbb{U}}, \preceq_{XY} \rangle$.

Part III

PRACTICAL CONTRIBUTIONS

10 PRACTICAL CONTRIBUTION OF NEW INTERVAL ENTROPY ON VIDEO STREAMING RATING

In this chapter, we present a case study involving the interval entropy methodology developed throughout this work. For this, we consider an application in the area of network traffic classification.

The dynamicity of current networks and the similarity between the protocols used, associated with techniques such as cryptography for data privacy and oscillations in the operational conditions of networks, are relevant challenges for classifying network traffic in a classical way.

Network traffic classification is the process of identifying specific applications or activities by matching them with network traffic. This task is essential for network management and security (Wang et al., 2017).

In (Monks, 2023) a hybrid approach was proposed, called FuzzyNetClass, for the classification of network traffic, focused on the current profile of computer network usage. This approach considers uncertainties generated by fluctuations in network resources of shared infrastructures, which are non-deterministic in nature.

By applying entropy to this approach, we enable the quantifying inaccurate or unknown information about its input and output data. Thus, based on the proposed methodology composed by the six methods presented in Section 8.3, the comparisons among the results through distinct admissible orders are performed, including the particular analysis considering the $\preceq_{\mathbf{A}}$-order, as introduced in this research work.

10.1 Main Concepts of FuzzyNetClass Approach

The FuzzyNetClass approach aims to contribute to the classification of traffic related to video streaming protocols, exploring the integration of inference systems based on interval-valued fuzzy logic and machine learning algorithms.

As can be seen in Figure 7, its architecture is divided into three modules: data insertion, network traffic classification and data extraction. In the data insertion stage, the network flow is captured using tools configured with selection of package types, for a specified period of time. From this, there is an extraction and selection of the attributes of the packets that were collected in the network.

The traffic classification step networking can occur through the fuzzy approach or through a hybrid approach that uses machine learning and fuzzy logic techniques. Also including the phases of fuzzification, rule base, inference and finally defuzzification through a type reducer.

Finally, the data extraction step that goes through the selection of the output type, which can be of three options: crisp data, linguistic terms and interval data. In the interval data option, it is possible to calculate the entropy, and thus, the contribution step of our study with this approach is located here.

In this case study we consider the Datasets captured from real network traffic, from the Federal University of Pelotas. It is important to mention that this capture was only possible thanks to the support of servers in the network sector from the university.

From the traffic capture, on different days and periods, the FuzzyNetClass approach was able to identify the protocols that circulated through the network, especially the video streaming ones, which were of interest to such an application. Four datasets were generated, each with eleven linguistic terms. These linguistic terms characterize the nature of the identification proposed by the approach and have five linguistic terms that correspond to the interval fuzzy sets "very low", "low", "below reasonable", "reasonable" and "high", which identified the amount of chance the analyzed protocol has to be dealing with video.

Applying the different methods of interval entropy, in the input and output characteristics, we obtained results that validate the promising proposal of the hybrid approach FuzzyNetClass. The modeling of this identification generates output data that characterize three sets of interval values "low", "average" and "high".

The proposed methodologies are characterized by precision in the results obtained, by using an interval as a response. For purposes of comparison between the methods, we used admissible orders.

10.2 Application of Theoretical Contributions on Video Streaming Traffic Rating

For the application of the distinct theoretical constructions for width-based interval-valued entropy, and of the proposed methods from these constructions, this work shows the information evaluation provided by the fuzzy controller of the FuzzyNetClass approach. This approach considers the classification of traffic related to video streaming, exploring the integration of inference systems based on interval-valued fuzzy logic and machine learning algorithms.

They are applied considering admissible interleaving $\preceq_{\mathbf{A}}$-order and the well-known

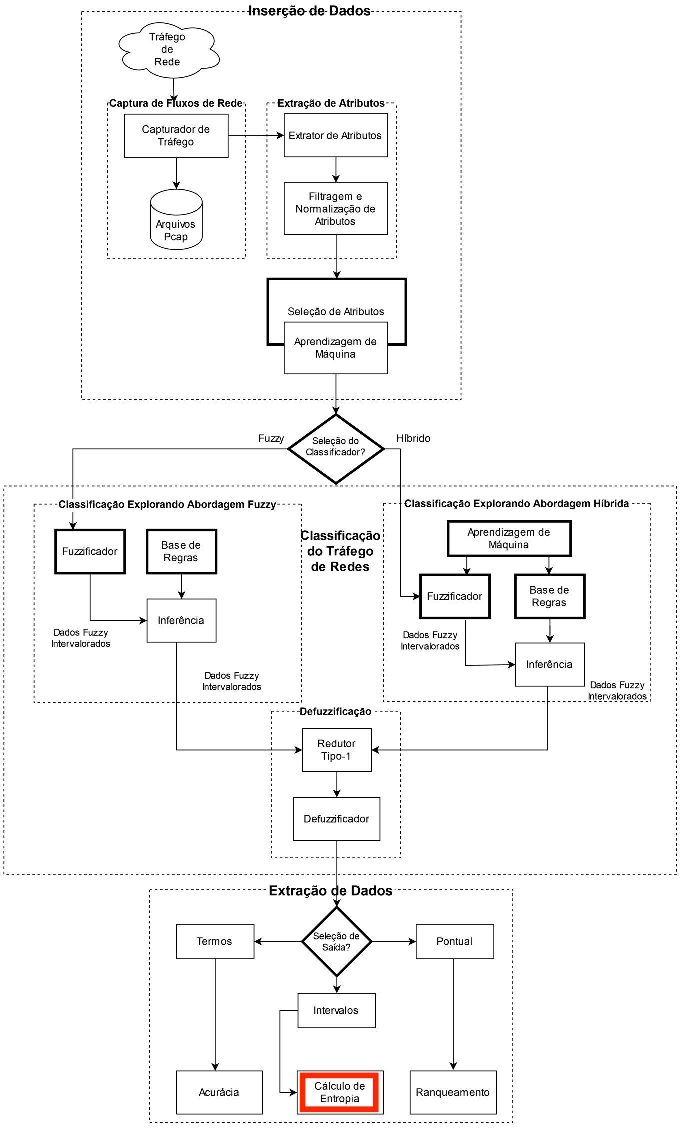

Figure 7 – FuzzyNetClass Architecture

Xu and Yager $\preceq_{XY}$-order. The IvE definitions are presented in two approaches, both based on a fuzzy entropy E w.r.t. the strong fuzzy negation N with equilibrium point on $\langle[0,1],\leq\rangle$.

Thus, in order to evaluate the contributions of entropy, six case studies were discussed, which considered four datasets conceived from real captures of network traffic.

The results obtained were promising and point to the continuation of study and research efforts on the subject of width-based interval entropy.

10.2.1 ω_A-IvE Applied to Video Streaming Traffic Classification

The methodology ω_A-IvE (width-based interval fuzzy entropy) proposed is now applied in the interpretation of (possible disorganized and/or discrepant) information related to output/input data in the FuzzyNetClass.

The considered approach is directed to the current profile of the use of computer networks, considering the uncertainties generated by fluctuations in network flow behavior and resources of the infrastructures shared, which are set as non-deterministic. Such conditions increase the complexity of accurate results to identify, in a network, the video protocols (on demand and live streaming) and differentiate them from of other protocols in network traffic.

For that, the analysis of interval fuzzy entropy measures and their comparisons based on admissible orders demand interpretation of input/output IvFS which are modeling the fuzzy control in the FuzzyNetClass. The eleven selected attributes or Linguistic Variables (LV), all of them normalized to the range of $[0, 10]$, each with 05 Linguistic Terms (LT) as "VeryLow"(V), "Low"(L), "BellowReasonable"(B), "Reasonable"(R) and "High"(H) comprising 55 IvFS to modeling input data as presented in Table 13.

Moreover, three linguistic terms are associated with the output linguistic variable Video:
(i) "LowVideo"(Lv), establishing the chance is low that the design control will detect a characterization as a video streaming in the flows fuzzy classifier;
(ii) "AverageVideo" (Av): presenting a moderate chance of detection of a potential characterization of video streaming by the design control;
(iii) "HighVideo"(Hv), considering the chance is high that the design control will detect a potential characterization of a video streaming in the flows fuzzy classifier.

Table 13 – Selected Input Attributes in the FuzzyNetClass Approach

Attribute	**Description**
Fwd Packet Length Mean	Mean size of packet in forward direction
Fwd Packet Length Std	Standard deviation Fwd Packet Length Mean
Bwd Packet Length Mean	Mean size of packet in backward direction
Bwd Packet Length Std	Standard deviation of Bwd Packet Length Mean
Flow IAT Mean	Mean value of the inter-arrival time of the flow
Flow IAT Std	Standard deviation of Flow IAT Mean
Fwd IAT Mean	Inter-arrival mean time of packets in forward direction
Fwd IAT Std	Standard deviation of Fwd IAT Mean
Bwd IAT Mean	Inter-arrival mean time of packets in backward direction
PLM	Packet Length Mean
PLS	Standard Deviation of Packet Length Mean

In the input attributes extraction, the selection algorithms were supervised by CREI/UFPEL [1] experts from modeling, through development and information validation of 04 elected datasets ($\mathbf{D_A}$, $\mathbf{D_B}$, $\mathbf{D_C}$, $\mathbf{D_D}$) supporting video streaming traffic classification. The validation is performed via such datasets considering the ω_A-IvE methods analysis, as a metric to measure the disordered information through interval-valued

[1] `https://institucional.ufpel.edu.br/unidades/id/944`

fuzzy data.

FuzzyNetClass's interval-valued fuzzy inference system considered lower and upper trapezoidal membership functions. The inference process applies the Mamdani method, considering a rule base with "AND" logical connective which applies triangular norms, totalizing 140 distinct rules. The defuzzification step is performed by two Type-1 Reductors, named Centroid (C) and Center of Sets (CoS) taking the fuzzy outputs and converting them to a single or crisp output value. In addition, the FuzzyNetClass controller can also identify the linguistic terms, and the membership degrees and also enable a comparison over the output video data by the admissibility of $\preceq$-order relations.

10.2.2 ω_A-IvE Methods Interpreting Input Information on IvFS

In this section, the information related to the fuzzification step in the FuzzyNetClass approach, considering the proposed width-based interval entropy methods.

For the implementation of each of the ω_A-IvE methods, we use the GNU Octave [2] tool, which is free software under the terms of the GPL license. The code files used in this Thesis, referring to the ω_A-IvE methodology, are available in a public repository [3].

[2] https://octave.org/
[3] https://github.com/Lidicostas/wa-IvFE

Table 14 – Entropy Measures for Input Data

		Very Low		Low		Below Reasonable		Reasonable		High	
D	**M**	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
A	1	0.415077	0.44961	0.283394	0.30886	0.075666	0.079628	0.015045	0.015418	**0.006161**	**0.00621**
	2	0.062799	0.08345	0.042176	0.04521	0.066017	0.066017	0.03709	0.03709	**0.021140**	**0.02114**
	3	0.058216	0.116431	0.052931	0.105862	0.050575	0.101149	0.027054	0.054107	**0.014252**	**0.028505**
	4	**0.380804**	**0.419014**	0.564541	0.58641	0.773876	0.804021	0.886776	0.903371	0.845328	0.856192
	5	**0.38463**	**0.494215**	0.564245	0.643398	0.778492	0.842255	0.890245	0.924544	0.848242	0.869791
	6	0.039614	0.116431	0.023416	0.105862	0.051394	0.101149	0.029913	0.054107	**0.020529**	**0.028505**
B	1	0.338153	0.36205	0.27848	0.29944	0.093177	0.097558	0.008794	0.009045	**0.001483**	**0.001496**
	2	0.239899	0.2916	0.258455	0.299292	0.269885	0.298825	0.270702	0.285383	**0.251019**	**0.258834**
	3	0.051391	0.102783	0.045658	0.091316	0.046834	0.093669	0.029064	0.058128	**0.015324**	**0.030649**
	4	**0.420155**	**0.455952**	0.619302	0.637783	0.788813	0.817528	0.867324	0.884849	0.797547	0.808605
	5	**0.239074**	**0.289154**	0.27326	0.312343	0.299473	0.33018	0.306201	0.322604	0.282103	0.290993
	6	0.036786	0.102783	0.021306	0.091316	0.048767	0.093669	0.030433	0.058128	**0.022058**	**0.030648**
C	1	0.326598	0.3522	0.256412	0.27705	0.092595	0.096987	0.023763	0.024385	**0.008630**	**0.008699**
	2	0.058056	0.081577	0.037759	0.040826	0.061849	0.061849	0.035627	0.035627	**0.021549**	**0.021549**
	3	0.055737	0.111473	0.047446	0.094891	0.046387	0.092774	0.027044	0.054087	**0.014266**	**0.028533**
	4	**0.399337**	**0.435907**	0.593377	0.614861	0.788625	0.816873	0.878551	0.895142	0.820577	0.830599
	5	0.200436	0.245628	0.22317	0.260453	0.235254	0.263733	0.227947	0.243771	**0.205008**	**0.212646**
	6	0.038783	0.111471	0.020397	0.094891	0.047484	0.092774	0.028599	0.054087	**0.020579**	**0.028533**
D	1	0.362322	0.39159	0.232535	0.25246	0.077767	0.081537	0.019873	0.020307	**0.005583**	**0.005619**
	2	0.061877	0.087209	0.041909	0.043888	0.05784	0.05784	0.030139	0.030139	**0.017923**	**0.017923**
	3	0.057914	0.115829	0.050667	0.101334	0.045253	0.090506	0.022814	0.045628	**0.011495**	**0.02299**
	4	**0.385988**	**0.425088**	0.630786	0.652345	0.792621	0.819235	0.89943	0.913396	0.826297	0.834654
	5	0.180348	0.223705	0.195979	0.229483	0.208586	0.233434	0.202202	0.215882	**0.179451**	**0.185672**
	6	0.041086	0.115828	0.022431	0.101334	0.04427	0.090506	0.024286	0.045628	**0.016666**	**0.022989**

• **D**: Dataset • **M**: ω_A-IvE Method

In Table 14, for each of the selected datasets ($\mathbf{D_A}$, $\mathbf{D_B}$, $\mathbf{D_C}$, $\mathbf{D_D}$), are provided the results from average (arithmetic means) of the 11 input attributes, as presented in Table 13, and related to ω_A-IvE of the five IvFS defined by the linguistic variables.

The rows from 1 to 6, describing the results of the methodologies for calculating the width-based interval fuzzy entropy. They are showing the results obtained as the arithmetic mean of the 11 selected input attributes, which were performed over the lower and upper bond of each IvFS related to linguistic variables ("VeryLow", "Low", "BelowReasonable", "Reasonable", "High").

The data are given by the arithmetic mean performed overall 11 attributes (linguistic variables) related to the linguistic variable "High", giving a relevant interpretation for the high compatibility of each network flow to the IvFS defined by this variable H. So, the input IvFS H achieves the least entropy measurements in four ω_A-IvE methods, and for all selected datasets. In contrast, the input IvFS identified by LV "VeryLow" achieves the least entropy measurements in two ω_A-IvE methods. These are detailed in the following comparisons based on admissible interleaving $\preceq_{\mathbf{A}}$-order:

I. Dataset $\mathbf{D_A}$

(1) Entropy Measures of **IvFS "High"(H)**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.006161, 0.00621]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.000049$;

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) = [0.021140, 0.02114]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H)) = 0.0$;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) = [0.014252, 0.028505])$ and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H)) = 0.014253$;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_H) = [0.020529, 0.028505]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_H)) = 0.007976$.

Moreover, $\mathbb{E}^N_\omega(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H)$.

(2) Entropy Measures of **IvFS "VeryLow"(V)**

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V) = [0.380804, 0.419014]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V)) = 0.03821$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_V) = [0.38463, 0.494215]$ and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_V)) = 0.109585$;

And, $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_V)$.

II. Dataset $\mathbf{D_B}$

(1) Entropy Measures of **IvFS "High"(H)**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.001483, 0.001496]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.000013$;

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) = [0.251019, 0.258834]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H)) = 0.007815$;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) = [0.015324, 0.030649])$ and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H)) = 0.015325$;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_H) = [0.022058, 0.030648]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_H)) = 0.008590$.

Moreover, $\mathbb{E}^N_\omega(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H)$.

(2) Entropy Measures of **IvFS "VeryLow"(V)**

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V) = [0.420155, 0.455952]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V)) = 0.035837$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_V) = [0.239074, 0.289154]$ and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_V)) = 0.05008$;

And, $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_V) \preceq \mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V)$.

III. Dataset $\mathbf{D_C}$

(1) Entropy Measures of **IvFS "High"(H)**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.008630, 0.008699]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.000069$;

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) = [0.021549, 0.021549]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H)) = 0.00$;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) = [0.014266, 0.028533])$ and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H)) = 0.014267$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_H) = [0.205008, 0.212646]$ and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_H)) = 0.007638$;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_H) = [0.020579, 0.028533]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_H)) = 0.007954$.

Moreover, $\mathbb{E}^N_\omega(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_H)$.

(2) Entropy Measures of **IvFS "VeryLow"(V)**

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V) = [0.399337, 0.435907]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V)) = 0.03657$.

IV. Dataset $\mathbf{D_D}$

(1) Entropy Measures of **IvFS "High"(H)**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.005583, 0.005619]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.000036$;

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) = [0.017923, 0.017923]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H)) = 0.0$;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) = [0.011495, 0.02299])$ and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H)) = 0.011495$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_H) = [0.179451, 0.185672]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_H)) = 0.006221$;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_H) = [0.016666, 0.022989]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_H)) = 0.006323$.

Moreover, $\mathbb{E}^N_\omega(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_H)$.

(2) Entropy Measures of **IvFS "VeryLow"(V)**

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V) = [0.385988, 0.425088]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_V)) = 0.0391$.

These results can also be graphically described based on bar graphs to compare and contrast different methods to measure IvE in input attributes on FuzzyNetClass, as presented in Figure 8, seen below.

Concluding the data analysis in Table 14, one can easily observe that Methods 04 and 05 show more sensibility, presenting the greatest intervals related to ω_A-IvE of IvFS w.r.t. the admissible pairwise $(\preceq_{\mathbf{A}}, \preceq_{\overleftarrow{\mathbf{A}}})$-order. In particular, for linguist term "Reasonable" it is detailed in the following results:

I. Dataset $\mathbf{D_A}$

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R) = [0.886776, 0.903371]$, and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R)) = 0.016595$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_R) = [0.890245, 0.924544]$, and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_R)) = 0.034299$.

II. Dataset $\mathbf{D_B}$

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R) = [0.867324, 0.884849]$, and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R)) = 0.017525$.

III. Dataset $\mathbf{D_C}$

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R) = [0.878551, 0.895142]$, and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R)) = 0.016591$.

IV. Dataset $\mathbf{D_D}$

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R) = [0.89943, 0.913396]$, and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_R)) = 0.013966$.

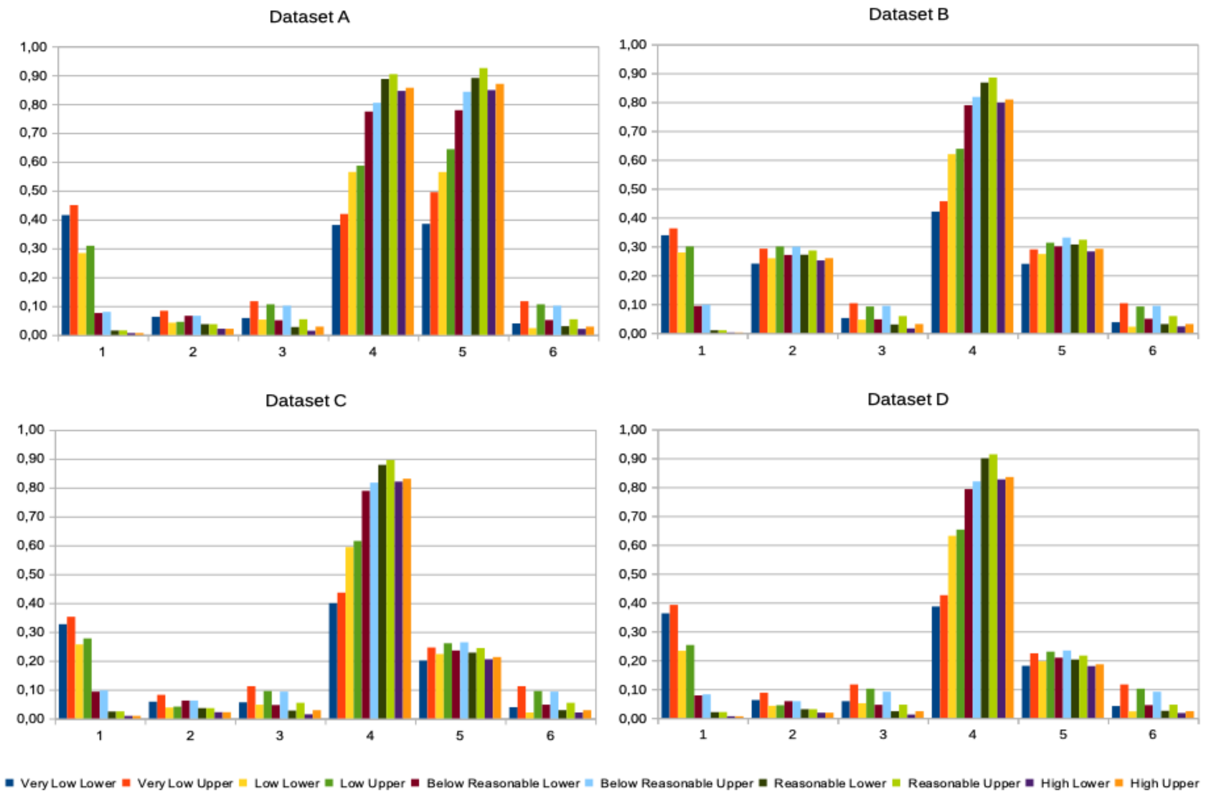

Figure 8 – Entropy Measures for Input Data

By flexibility of our algorithmic proposal, the above analysis suggests a reevaluation of M4 and M5 methodologies, reloading new parameters and operators (as aggregation, negation, and restricted equivalence functions) in the definition of ω_A-IvE, considering the decision-making for this application directed to video streaming traffic classification. So, it can improve the M4 and M5 entropy measure estimation by comparing these to the results achieved by the others.

10.2.3 ω_A-IvE Methods Interpreting Output Information of IvFS

In this section, the information in the defuzzification step of the FuzzyNetClass is analyzed considering the proposed width-based interval entropy methods. To do so, Table 15 presents the average related to the entropy measures for output IvFS, considering in the defuzzification step two type-1 reduction techniques (T1R), the Center of Sets (CoS), and the Centroid (C).

Table 15 – Entropy Measures for Output IvFS - Center of Sets and Centroid

		Center of Sets						Centroid					
D	**M**	Low		Average		High		Low		Average		High	
		Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper	Lower	Upper
A	1	0.4294	0.4496	0.2734	0.3089	**0.0741**	**0.0796**	0.4269	0.4496	0.2864	0.3089	**0.0770**	**0.0796**
	2	**0.1385**	**0.1385**	0.1698	0.2807	0.1826	0.1926	0.1674	0.1674	**0.0915**	**0.1510**	0.1760	0.1760
	3	**0.0859**	**0.1718**	0.1819	0.3639	0.1172	0.2343	0.1034	0.2067	0.1000	0.2001	**0.0898**	**0.1797**
	4	0.7522	0.8138	**0.5329**	**0.6908**	0.6048	0.6764	0.7402	0.8221	**0.3909**	**0.4638**	0.8778	0.9370
	5	0.7568	0.8542	0.5567	0.8000	**0.5954**	**0.7404**	0.7519	0.8745	**0.4456**	**0.6291**	0.8908	0.9669
	6	**0.1268**	**0.1718**	0.2490	0.3639	0.1650	0.2343	0.1561	0.2067	0.1273	0.2001	**0.1462**	**0.1797**
B	1	0.3462	0.3621	0.2738	0.2994	**0.0919**	**0.0976**	0.3433	0.3621	0.2772	0.2994	**0.0933**	**0.0976**
	2	0.1742	0.1742	0.2362	0.2362	**0.2206**	**0.2206**	0.1888	0.1888	**0.1668**	**0.1773**	0.2531	0.2531
	3	**0.0998**	**0.1996**	0.1394	0.2787	0.1233	0.2466	0.1121	0.2242	0.1156	0.2311	**0.1276**	**0.2553**
	4	0.7110	0.7630	**0.6725**	**0.7636**	0.7017	0.7722	0.6777	0.7499	**0.4929**	**0.5975**	0.8463	0.9265
	5	0.7161	0.8247	0.6990	0.8627	**0.6850**	**0.8443**	0.6788	0.8132	**0.5225**	**0.7163**	0.8670	0.9703
	6	**0.1559**	**0.1996**	0.1931	0.2787	0.1887	0.2466	0.1724	0.2242	**0.1569**	**0.2311**	0.2118	0.2553
C	1	0.3393	0.3522	0.2503	0.2771	**0.0899**	**0.0970**	0.3366	0.3522	0.2554	0.2771	**0.0926**	**0.0970**
	2	**0.1361**	**0.1361**	0.2002	0.2324	0.2116	0.2317	0.1537	0.1537	**0.1364**	**0.1770**	0.2116	0.2116
	3	**0.0806**	**0.1611**	0.1508	0.3016	0.1345	0.2690	**0.0934**	**0.1868**	0.1132	0.2265	0.1132	0.2265
	4	0.7847	0.8287	**0.5853**	**0.7086**	0.6253	0.7081	0.7471	0.8107	**0.4285**	**0.5100**	0.8183	0.8984
	5	0.7905	0.8808	0.6118	0.8151	**0.6130**	**0.7782**	0.7503	0.8662	**0.4811**	**0.6567**	0.8341	0.9396
	6	**0.1245**	**0.1611**	0.2052	0.3016	0.1964	0.2690	**0.1425**	**0.1868**	0.1483	0.2265	0.1810	0.2265
D	1	0.3725	0.3916	0.2203	0.2525	**0.0763**	**0.0815**	0.3707	0.3916	0.2306	0.2525	**0.0795**	**0.0815**
	2	0.1431	0.1431	0.1879	0.2926	**0.1323**	**0.1323**	0.1690	0.1690	0.1543	0.2149	**0.1074**	**0.1074**
	3	0.0900	0.1799	0.1906	0.3811	**0.0881**	**0.1762**	0.1061	0.2123	0.1326	0.2651	**0.0588**	**0.1175**
	4	0.7276	0.7762	**0.4993**	**0.6643**	0.6440	0.6941	0.6883	0.7656	**0.4023**	**0.4934**	0.8879	0.9282
	5	0.7354	0.8377	**0.5253**	**0.7962**	0.6375	0.7411	0.6931	0.8306	**0.4688**	**0.6755**	0.8945	0.9510
	6	0.1311	0.1799	0.2537	0.3811	**0.1114**	**0.1762**	0.1590	0.2123	0.1786	0.2651	**0.0920**	**0.1175**

• **D**: Dataset • **M**: ω_A-IvE Methodology

The bold intervals highlight the best results for each of the width-based interval entropy methods, considering the four datasets. It is immediate that the number of such intervals is much more frequent in the two last columns (mapping results from linguistic terms “Average” and “High”) in the Centroid case. These are discussed in the following:

I. Dataset $\mathbf{D_A}$

(1) Entropy Measures in **CoS/T1R**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.0741, 0.0796]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.0055$ in CoS /T1R;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L) = [0.0859, 0.1718]$) and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L)) = 0.0859$ in CoS/T1R;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_L) = [0.1268, 0.1718]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_L)) = 0.045$ in CoS/T1R.

Thus, we have that $\mathbb{E}^N_\omega(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_H)$.

(2) Entropy Measures in **C/T1R**

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_A) = [0.0915, 0.1510]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_A)) = 0.0595$ in C/T1R;

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A) = [0.3909, 0.4638]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A)) = 0.0729$ in C/T1R.

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A) = [0.4456, 0.6291]$ and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A)) = 0.1835$ in C/T1R.

Thus, we have that $\mathbb{E}^N_{\mathbf{A}} \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega} \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{R},\omega}$.

II. Dataset $\mathbf{D_B}$

(1) Entropy Measures in **CoS/T1R**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.0919, 0.0976]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.0057$;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L) = [0.0998, 0.1996]$) and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L)) = 0.0998$;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_L) = [0.1559, 0.1996]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_L)) = 0.0437$.

Thus, we have that $\mathbb{E}^N_\omega(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_L)$.

(2) Entropy Measures in **C/T1R**

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_A) = [0.1668, 0.1773]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_A)) = 0.0105$;

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A) = [0.4929, 0.5975]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A)) = 0.1046$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A) = [0.5225, 0.7163]$ and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A)) = 0.1938$;

Thus, we have that $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_A) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A)$

III. Dataset $\mathbf{D_C}$

(1) Entropy Measures in **CoS/T1R**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.0899, 0.0970]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.0071$;

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_L) = [0.1361, 0.1361]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_L)) = 0.0$;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L) = [0.0806, 0.1611]$) and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L)) = 0.0805$;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_L) = [0.1245, 0.1611]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_L)) = 0.0366$.

Thus, we have that $\mathbb{E}^N_\omega(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_L) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_L) \preceq_{\mathbf{A}} \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_L)$.

(2) Entropy Measures in **C/T1R**

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A) = [0.4285, 0.5100]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A)) = 0.0815$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A) = [0.4811, 0.6567]$ and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A)) = 0.1756$.

Thus, we have that $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A)$

IV. Dataset $\mathbf{D_D}$

(1) Entropy Measures in **CoS/T1R**

$\mathbf{M}1$: $\mathbb{E}^N_\omega(\mathbb{A}_H) = [0.0763, 0.0815]$ and $\omega(\mathbb{E}^N_\omega(\mathbb{A}_H)) = 0.0052$;

(2) Entropy Measures in **C/T1R**

$\mathbf{M}2$: $\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) = [0.1074, 0.1074]$ and $\omega(\mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H)) = 0.0$;

$\mathbf{M}3$: $\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) = [0.0588, 0.1175]$) and $\omega(\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H)) = 0.0587$;

$\mathbf{M}4$: $\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A) = [0.4023, 0.4934]$ and $\omega(\mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A)) = 0.0911$;

$\mathbf{M}5$: $\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A) = [0.4688, 0.6755]$ and $\omega(\mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A)) = 0.2067$;

$\mathbf{M}6$: $\mathbb{E}^p_{IV}(\mathbb{A}_H) = [0.0920, 0.1175]$ and $\omega(\mathbb{E}^p_{IV}(\mathbb{A}_H)) = 0.0255$.

Thus, we have the following comparison:

$\mathbb{E}_{\mathbb{S},\omega}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^p_{IV}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}^N_{\mathbf{A}}(\mathbb{A}_H) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{S}_{\mathbf{A}},\omega}(\mathbb{A}_A) \preceq_{\mathbf{A}} \mathbb{E}_{\mathbb{R},\omega}(\mathbb{A}_A)$

Analogously, the results exploring ω_A-IvE applied to input of IvFS from the fuzzification step, the greatest ω_A-IvE of IvFS in the defuzzification step addressing the output data before the action of type-1 reduction is related to $M4$ and $M5$ methodologies, expressing the major sensibility in FuzzyNetClass application. These methodologies based on IvREF methods increment in the analysis of disorganized information, when they are compared to the other ones. This analysis happened in the four datasets, as shown in the bar graphs in Figure 9.

For the IvFS associated with the linguistic terms "AverageVideo" and "HighVideo", the comparison using an admissible interleaving order shows that in both cases, the results indicate lower IVE for information in the defuzzification process via the Centroid methodology compared to the Center of Sets reduction methodologies. The corresponding diameter was also smaller in these cases. Therefore, this entropy measure indicates that for the linguistic variables "Average" and "High," there is less disorganization of information and less imprecision regarding the output IvFS.

Therefore, for the FuzzyNetClass hybrid approach, the output data for IvFS related to the linguistic variables "Average" and "High" classes show a lower upper bound (and diameter) for interval entropy performed by M1, M2, M3, and M6. Thus, using these four methods, the FuzzyNetClass demonstrates proficiency in information classification, encompassing both the "On Demand" and "Live Streaming" video streams.

10.3 Summary

In this chapter, we can check the application of the methodology ω_A-IvE (width-based interval fuzzy entropy) to a real inference system. The proposed methods promote distinct interpretations of (possibly disorganized and/or discrepant) information related to output/input data in the FuzzyNetClass – which is a hybrid approach to net-

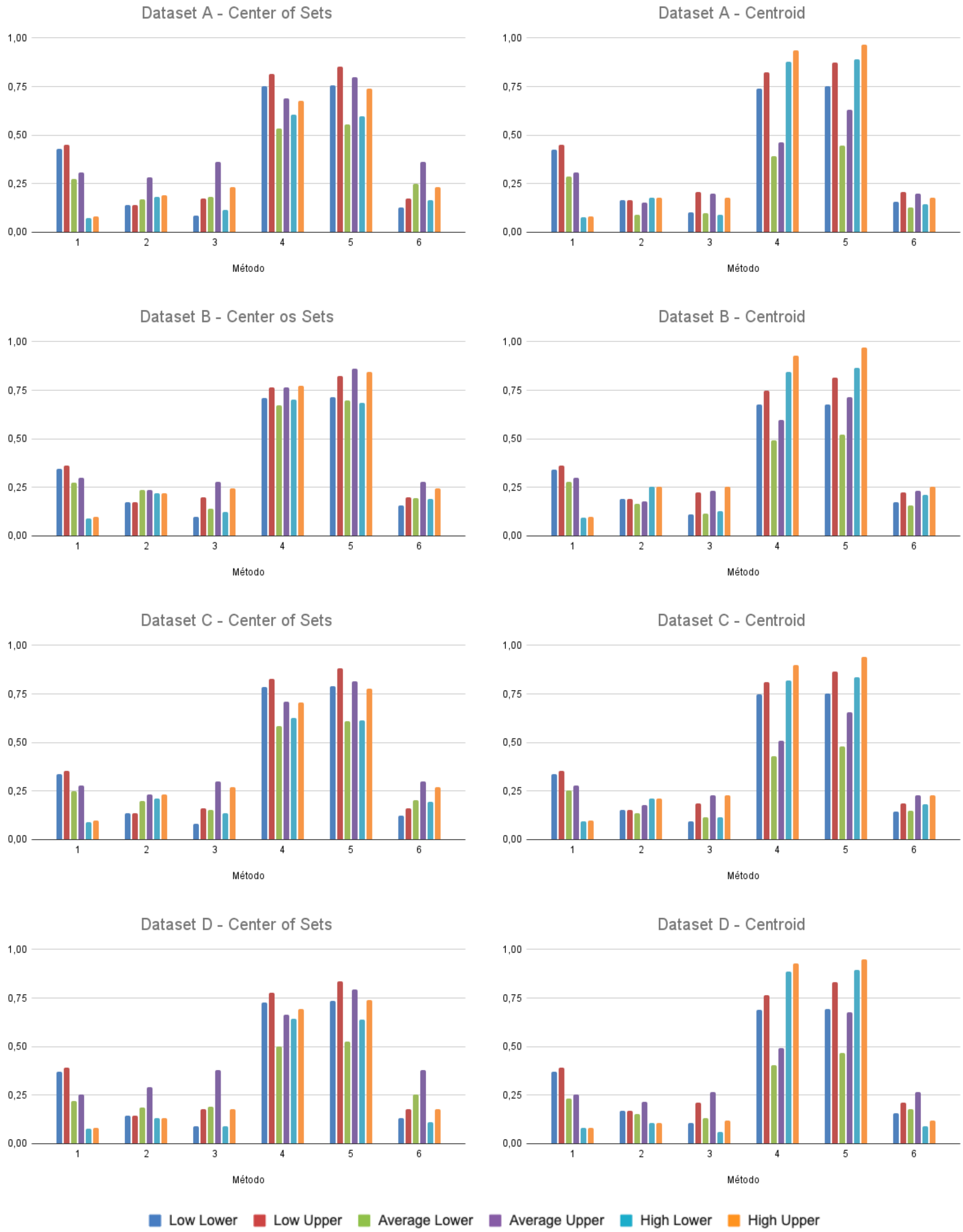

Figure 9 – Entropy Analysis of Output Attribute

work traffic classification integrating fuzzy approximate thinking and machine learning techniques.

This practical contribution of this thesis is applied to video streaming ratings. For that, the main concepts of the FuzzyNetClass approach are described and then, the application of contributions on video streaming traffic rating is discussed.

The algorithm expression for the methodology based on $\omega_{\mathbf{A}}$-IvE is described, considering the video streaming traffic classification. The five $\omega_{\mathbf{A}}$-IvE methods introduced in this work and another one previously presented in the literature are implemented and evaluated by the selected databases. The $\omega_{\mathbf{A}}$-IvE methods interpreting output information of IvFS are considered to analyze the results obtained in terms of graphical presentation and discussion on the data preservation w.r.t. the relation of input/output interval diameters.

11 CONCLUSION

This chapter describes the main contributions of this work and also points out possible further work.

11.1 Main Results

This work contributed to distinct approaches, extending formal concepts and methodologies to obtain width-based interval fuzzy entropy and an A-IvIFE, including applying these theoretical results in the evaluation of video information provided by the FuzzyNetClass approach.

11.1.1 Results on Theoretical Contributions

A. Theoretical Contributions on Interval Fuzzy Entropy Measures

This section introduces the theoretical studies described in this work starting with the notion of just one injective and increasing function $\mathbf{A} : \mathbb{U} \rightarrow U$ w.r.t the corresponding product order on $\mathbb{U}$ and the usual order on U to generate admissible orders $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$.

Thus, the admissible interleaving functions $\mathbf{A}, \overleftarrow{\mathbf{A}} : \mathbb{U} \rightarrow U$, respectively named as right and left Decimal-Digit Interleaving (DDI) are introduced, as particular injective and increasing functions. This study includes examples and propositions discussing their main properties. Moreover, the two reverse constructions of such DDI-based admissible interleaving functions indicated as $\mathbf{A}^{(-1)}$ and $\overleftarrow{\mathbf{A}}^{(-1)}$, are also analyzed, underlying the study of (anti-) monotonicity of admissible interleaving interval-valued fuzzy connectives on $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$.

The main results extend the definition of fuzzy connectives on $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$, considering admissible interleaving negations generated by strong fuzzy negations with equilibrium point, as N_S and N_e, and also width-based interval-valued operators on $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$, as aggregation functions, IvREF and IvRDF functions.

In sequence, important concepts on width-based interval fuzzy entropy are presented considering the set $\mathcal{A}_{\mathbb{U}}$ of all interval-valued fuzzy sets defined on the finite and non-empty universe χ.

The relevant discussion of the main conditions to define width-based interval-valued entropy ($\omega_{\mathbf{A}}$-IvE) measures on $\langle \mathbb{U}, \preceq_{\mathbf{A}} \rangle$ takes into account the DDI-based admissible interleaving $\preceq_{\mathbf{A}}$-order.

- The first approaches consider the IvE on $\langle \mathbb{U}, \preceq \rangle$, which is generated by an averaging aggregation function and width-based functions on $\langle U, \leq \rangle$, as an interval-valued fuzzy negation with equilibrium point and an average aggregation. In particular, for such proposed IvE, the text promotes illustrative examples for the DDI-based admissible interleaving $\preceq_{\mathbf{A}}$-order and the Xu-Yager $\preceq_{XY}$-order.

- Then, in sequence, the second approach to generate width-based interval-valued entropy (IvE) on $\langle \mathbb{U}, \preceq \rangle$ considers the width-based restricted equivalence functions and strong fuzzy negation with equilibrium point, also including a width-based interval-valued average aggregation on $\langle \mathbb{U}, \preceq \rangle$. Thus, three other definitions are proposed by varying the aggregation operators. To illustrate the construction, the DDI-based admissible interleaving $\preceq_{\mathbf{A}}$-order and the Xu-Yager $\preceq_{XY}$-order are considered.

Each one of the described IvE definitions gives us 05 methodologies to obtain width-based interval-valued entropy (ω-IvE) on $\langle \mathbb{U}, \preceq \rangle$. Moreover, to compare the results of the proposal application, an additional method reports the definition of ω-IvE as presented in (Takáč et al., 2019).

B. Theoretical Contributions on Atanassov's Interval-valued Intuitionistic Fuzzy Entropy

Making use of A-IvIFL and the admissible orders on $\langle \tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}} \rangle$, this work consolidate the following both approaches:

- Synthesizing the notion of admissible orders, the general concept of the generalized A-GIvIFIx associated with a strong interval-valued fuzzy negation is characterized in terms of interval-valued fuzzy implication operators as a construction method to model hesitation in A-IvIFS;

- Promoting the interval extension of Atanassov's intuitionistic fuzzy entropy as proposed in (Bustince et al., 2011), which considers aggregation the generalized Atanassov's Intuitionistic Fuzzy Index related to admissible orders, in particular considering the well-known Xu and Yager $\preceq_{XY}$-order.

Moreover, this thesis also consolidates a more general concept of the generalized Atanassov's interval-valued intuitionistic fuzzy index associated with a strong interval-valued intuitionistic fuzzy negation. And, fuzzy implication operators are characterized to construct methodologies taking into account the hesitation and imprecision modeling in an A-IvIFS, by using total orders. Thus, the application of admissible linear order provides results in comparison to the methodology for building an interval fuzzy entropy on $\tilde{\mathbb{U}}$.

11.1.2 Application of Theoretical Results on Video Streaming Traffic Rating

For the application of the distinct theoretical constructions for width-based interval-valued entropy, and of the proposed methods from these constructions, this work shows the information evaluation provided by the fuzzy controller of the FuzzyNetClass approach. This approach considers the classification of traffic related to video streaming, exploring the integration of inference systems based on interval-valued fuzzy logic and machine learning algorithms.

They are applied considering the admissible interleaving $\preceq_{\mathbf{A}}$-order and the well-known Xu and Yager's $\preceq_{XY}$-order. The IvE definitions are presented in two approaches, both are based on a fuzzy entropy E w.r.t. a strong fuzzy negation N with equilibrium point on $\langle U, \leq \rangle$.

In this perspective, the FuzzyNetClass approach integrates two research areas: (i) Machine Learning, considering algorithms that contribute to the classification performed; and (ii) Fuzzy Control Systems, preserving for the specialists involved aspects related to its interpretability, e.g., the relation between cause and effect observed on such system.

Among the contributions of the FuzzyNetClass approach, the following stand out Hybrid Classification Algorithms, based on the knowledge of specialists and on the exploration of interval-valued fuzzy logic. However, such an approach also explores the potential to provide more reliable and realistic results and the design of optimization mechanisms that come from the integration between Machine Learning techniques and Fuzzy Inference.

In sequence, to evaluate the contributions of entropy, 06 case studies were discussed, which considered 04 datasets conceived from real captures of network traffic. The results obtained were promising and point to the continuation of study and research efforts on the subject of width-based interval-valued entropy.

11.1.3 Presentation and Publication of Main Results

The main publications obtained from the studies developed throughout this work are shown in Table 16.

Table 16 – Publications

Publication	Title	Year
NAFIPS	Interval version of Generalized Atanassov's Intuitionistic Fuzzy Index	2018
WEIT	Generalized Atanassov's Intuitionistic Fuzzy Index: Interpreting Hesitance, Favour and Against degrees	2019
FUZZIEEE	Interval Extension of the Generalized Atanassov's Intuitionistic Fuzzy Index using Admissible Orders	2019
EUSFLAT	Extending representability on the set of intervals endowed with admissible orders for the construction of interval-valued fuzzy operators	2021
International Journal of Approximation Reasonable	$\omega_{\mathbf{A}}$-IvE Methodology: Admissible Interleaving Entropy Methods Applied to Video Streaming Traffic Classification (in revision)	2023

11.2 Further Work

Further work considers the extension of our results related to other properties verified by the generalized Atanassov's interval-valued intuitionistic fuzzy index and interval extension Atanassov's intuitionistic fuzzy entropy. Besides, it also takes into account others classes of admissible linear orders to compare the results of the interval entropy.

We also intend to handle both problems, focusing particularly on how other aggregations can be used to obtain interval-valued intuitionistic fuzzy entropy based on generalized Atanassov's interval-valued intuitionistic fuzzy index, for instance, Choquet's integral (Choquet, 1954) allows us to define many of the most usual aggregation functions.

Due to the relevance of the theoretical methods to calculate the interval entropy, we leave for future work the deeper study of the negations and aggregations w.r.t. the admissibility provided by $\preceq_{\mathbf{A}}$-order, i.e., analyzing the conditions under which the methodology and illustrative examples can improve the inference by enabling a non-restrictive comparison of data information supporting decision-making systems.

Another proposal for extending the research themes is to formalize the study of total orders for interval data based on pairs of order relations, modeling the comparison of binary connectives presenting increasing monotone behavior in one argument and, decreasing monotone behavior in the other, e.g, the interval-valued fuzzy (co)implications.

Application of the methodology to analyze/compare data information in applied and theoretical research, e.g., those in development on LUPS/UFPEL [1] research group and

[1] https://wp.ufpel.edu.br/lups/

mainly addressing:

- Medical Diagnosis as clinic deterioration and/or early warning for patients in intensive care units, where hybrid approaches are considered to estimate the situation based on techniques from computational intelligence;

- Resource Discovery and Classification in IoT considering EXEHDA-Resource Ranking, a proposal stands out in IoT resource classification, exploring three approaches: (i) initial selection of resources with MCDA algorithm; (ii) pre-classification of newly discovered resources with machine learning; and (iii) treatment of uncertainty in preference processing using Interval-valued Fuzzy Logic.

- Dynamic Consolidation of the Virtual Machines in Cloud Computing, based on uncertainty provided by determining overloaded/underloaded physical machines, selection/allocation of virtual machines for migration. The entropy analysis impacts on resource management, and directly influence aspects of it‘s use, such as energy efficiency, SLA, and, consequently, QoS.

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